Minimal Surfaces in Euclidean Space

Lecture Notes, UAB, Spring 1992

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Contents

Chapter 1

Introduction

In this introductory lecture, I wish to present an overview of the questions and results which will be addressed during this term. The presentation is not rigorous and many details are omitted, but hopefully this chapter should serve as a road map.

1.1 Plateau's Problem

Our first goal is to prove the existence of a solution to Plateau's Problem [12, 4, 13]:

Problem 1.1 *Given a simple closed Jordan curve $\Gamma \subset \mathbb{R}^n$, which is the boundary of a finite area surface of disk type, find a surface of disk type S with boundary Γ, such that S has least area among all such surfaces.*

We would like to say that a surface is of *disk type* if it is homeomorphic to a disk, but as can be seen from simple examples, this would be too restrictive. Unless rather stringent conditions are put on the boundary curve, the solution may have self intersections [2]. Thus, we define S to be of disk type if it is the image of a smooth map:

$$X\colon D \to \mathbb{R}^n,$$

where $D = \{(u,v) : u^2 + v^2 < 1\} \subset \mathbb{R}^2$ is the unit disk in the plane, and $X = (X^1, \ldots, X^n)$ are Cartesian coordinates in n-dimensional Euclidean space. By *smooth*, we mean $X \in C^2(D) \cap C^0(\overline{D})$. Furthermore, we will require that ∇X is non-degenerate in D. This excludes the presence of so-called *branch points*. Finally for the boundary of $X(D)$ to be Γ, we will ask that $X|\partial D$ maps the circle ∂D continuously and monotonically onto Γ.

The *area* of the surface $X(D)$ is easily calculated to be:

$$A(X) = \int_D \sqrt{ef - g^2}, \tag{1.1}$$

where $e = |X_u|^2$, $f = |X_v|^2$, and $g = X_u \cdot X_v$. If the surface X is of least area, then a standard calculation shows that it satisfies the following system of partial differential equations, see Exercise 1.1:

$$\frac{\partial}{\partial u}\left(\frac{fX_u - gX_v}{\sqrt{ef-g^2}}\right) + \frac{\partial}{\partial v}\left(\frac{eX_v - gX_u}{\sqrt{ef-g^2}}\right) = 0. \tag{1.2}$$

A surface X satisfying Equations (1.2) is called a *minimal surface*, even though it may not be a surface of least area.

The area $A(X)$ is invariant under re-parameterizations, that is, if $\psi\colon D \to D$ is any smooth injective map of D onto itself, then

$$A(X) = A(X\circ\psi).$$

This leads to some difficulty, for if X is a solution of Problem 1.1, and ψ such a map, then clearly $X\circ\psi$ is another solution. Therefore, the solution of the problem, in terms of the maps X is not unique, and the problem is not well-posed. Another way to see this, is to note that Equations (1.2) are degenerate.

In order to overcome this difficulty, we recall how this is dealt with in the 1-dimensional problem: the *geodesic problem*. Consider a domain $\Omega \subset \mathbb{R}^n$ with a Riemannian metric g_{ij}, i.e. a smooth symmetric $n \times n$ matrix valued function defined on Ω. Given a curve, $\gamma = (\gamma^1, \ldots, \gamma^n)\colon I \to \Omega$ defined on a closed interval $I \subset \mathbb{R}$, the length of γ is:

$$L(\gamma) = \int_I |\dot\gamma|_g\,, \tag{1.3}$$

where $\dot\gamma = d\gamma/dt = (\dot\gamma^1, \ldots, \dot\gamma^n)$ is the tangent of γ, and its norm is taken with respect to the Riemannian metric g:

$$|\dot\gamma|_g^2 = g(\dot\gamma, \dot\gamma) = \sum_{i,j} g_{ij}\,\dot\gamma^i\,\dot\gamma^j. \tag{1.4}$$

The length $L(\gamma)$ is easily seen to be invariant under re-parameterizations, i.e. if ξ is any smooth monotonic map $\xi\colon I \to J$, then:

$$L(\gamma) = L(\gamma\circ\xi). \tag{1.5}$$

Therefore the same non-uniqueness problem arises when we try to minimize the length of a curve, say between two points p and $q \in \Omega$.

Define the *energy* of γ to be:

$$E(\gamma) = \frac{1}{2}\int_I |\dot\gamma|_g^2\,.$$

For simplicity, assume that $I = [0, 1]$. Then the Schwartz inequality gives

$$L(\gamma) \leq \sqrt{2E(\gamma)},$$

with equality if and only if $|\dot{\gamma}|_g$ is constant, i.e. γ is parameterized proportionally to arclength. Since every curve, in particular the solution, can be re-parameterized proportionally to arclength, we have $\inf L = \inf \sqrt{2E}$. Furthermore, minimizing E yields a solution parameterized proportionally to arclength. The solution is now unique, and the variational problem is well-posed. Note that without fixing I, the solution is invariant under a non-compact group of transformations, the linear maps, and any minimization scheme is doomed to failure.

The 2-dimensional analog of parameterizations proportional to arclength are the *conformal maps*. The map $X \colon D \to \mathbb{R}^n$ is said to be conformal if

$$|X_u|^2 = |X_v|^2, \qquad X_u \cdot X_v = 0. \tag{1.6}$$

If the map X is invertible in a neighborhood U of a point $p \in X(D)$, $X^{-1} \colon U \to D$ defines two functions (u, v) on U called *local isothermal* coordinates. The first result we need is the existence of local isothermal coordinates on any smooth surface S of disk type. This is a deep result, generally known as a uniformization theorem.

Define the energy of X by:

$$E(X) = \frac{1}{2} \int_D |\nabla X|^2, \tag{1.7}$$

where $|\nabla X|^2 = e + f$. The functional $E(X)$ is invariant under conformal maps of the domain D, see Exercise 1.3. If X minimizes E, then it is easily seen to satisfy the following system of linear elliptic partial differential equations:

$$\Delta X = 0. \tag{1.8}$$

A map X satisfying Equations (1.8) is said to be *harmonic*. Note that if X is harmonic and conformal, then it represents a minimal surface, i.e. satisfies (1.2).

Here, the theory of analytic functions comes into play, for as is well known, it follows from (1.8), that there exists a dual harmonic map Y, satisfying for each $j = 1, \ldots, n$:

$$\nabla X^j \cdot \nabla Y^j = 0. \tag{1.9}$$

Now, the function $\Psi = X + iY$ is an analytic function of the complex variable $\zeta = u + iv$, where $i = \sqrt{-1}$, the function $\Phi = \partial\Psi/\partial\zeta$ is also analytic, and the conformal relations amount to the condition

$$\sum_{j=1}^{n} \left(\Phi^j\right)^2 = 0. \tag{1.10}$$

This leads to the *Weierstrass representation* [10]. For simplicity take $n = 3$. Given two analytic functions $\psi(\zeta)$ and $\chi(\zeta)$ defined in an open set $\Omega \subset \mathbb{C}$, and $(x_0, y_0, z_0) \in \mathbb{R}^3$, then the surface described by:

$$\left.\begin{aligned} x &= x_0 + \Re \int \left(\psi^2 - \chi^2\right) d\zeta, \\ y &= y_0 + \Re \int i \left(\psi^2 + \chi^2\right) d\zeta, \\ z &= z_0 + \Re \int 2\psi\chi \; d\zeta, \end{aligned}\right\} \tag{1.11}$$

is a minimal surface, as is easily checked. Conversely, any minimal surface, can be locally represented by Equations (1.11), for some analytic functions ψ and χ.

From the inequality

$$\sqrt{ef - g^2} \leq \frac{1}{2}(e + f),$$

it follows that

$$A(X) \leq E(X),$$

with equality if and only if X is conformal, see (1.6). Thus, minimizing E is equivalent to minimizing $A(X)$ over all conformal parameterizations. However, even when we fix the parameter domain D, the solution is not unique. In fact, it is invariant under the Möbius group, the (non-compact) group of conformal transformations of the unit disk onto itself. Fortunately, this is a finite dimensional group, and imposing a finite number of auxiliary conditions, will yield a well-posed variational problem. This is usually done as follows. Fix three distinct points ω_j on the circle ∂D, and three point Q_j on Γ with the same order. Then, the map $X \colon D \to \mathbb{R}^n$ mapping ∂D onto Γ continuously and monotonically, is said to satisfy the *three points conditions*, if $X(\omega_j) = Q_j$. Any solution can be made to satisfy such a condition by composing it with a Möbius transformation.

Thus, we are led to study the following problem:

Problem 1.2 *Given a simple closed Jordan curve $\Gamma \subset \mathbb{R}^n$, find a map $X \colon D \to \mathbb{R}^n$, which maps ∂D to Γ continuously and monotonically, satisfies the three points condition, and such that $E(X)$ is the least among all such surfaces.*

1.2 Dirichlet's Principle

The solution of Problem 1.2 begins with the study of a simpler one, when a fixed monotonic parameterization $\gamma \colon \partial D \to \Gamma$ is given:

Problem 1.3 *Given a function $\gamma\colon \partial D \to \mathbb{R}^n$, find a map*

$$X\colon D \to \mathbb{R}^n, \quad X|\partial D = \gamma,$$

which minimizes $E(X)$ among all maps Y such that $Y|\partial D = \gamma$.

The existence of a solution to this problem is known as *Dirichlet's principle*, and was wrongly assumed to be true by Riemann for arbitrary data γ. That a solution does not necessarily exist, was first pointed out by Hadamard [3]. It is enough to consider the case $n = 1$. Let $f(\theta) \in C(\partial D)$, and expand f in a Fourier series:

$$f(\theta) \sim \sum_k a_k e^{ik\theta},$$

Then, u minimizes $E(u)$ over all functions v satisfying the boundary conditions, if and only if $\Delta u = 0$, see Theorem 1.3. Hence, for such a minimizing u,

$$u = \sum_k a_k r^{|k|} e^{ik\theta},$$

from which we calculate:

$$E_r(u) = \int_{D_r} |\nabla u|^2 = 2\pi \sum_k |k|\, |a_k|^2\, r^{2|k|},$$

where $D_r = \{(u, v) \in D; u^2 + v^2 < r\}$, and therefore:

$$E(u) = \lim_{r \to 1} E_r(u) = 2\pi \sum_k |k|\, |a_k|^2 .$$

However, one easily finds a function

$$f(\theta) = \sum_k \frac{\sin k!\theta}{k^2}$$

which is continuous, but for which the series

$$\sum_k \frac{k!}{k^2}$$

diverges. Thus, every v with $v|\partial D = f$ has $E(u) = \infty$. Dirichlet's principle does not hold with f as boundary values. Instead, one formulates:

Theorem 1.1 Dirichlet's Principle for the Disk
Let $\varphi \in C^0(\partial D)$, and define

$$\mathcal{H}_\varphi = \{v \in C^1(D) \cap C^0(\overline{D});\;\; v|\partial D = \varphi;\;\; E(v) < \infty\}$$

If $\mathcal{H}_\varphi \neq \emptyset$, then there exists $u \in \mathcal{H}_\varphi$ which minimizes E, i.e. $E(u) \leq E(v)$ for every $v \in \mathcal{H}_\varphi$. This solution $u \in C^\infty(D) \cap C^0(\overline{D})$, and satisfies in D:

$$\Delta u = 0. \tag{1.12}$$

Now, one sees the necessity of requiring in Problem 1.1 the existence of one surface of finite area spanning the given Jordan curve. To prove Theorem 1.1, one uses a standard variational scheme. First, enlarge the class of admissible maps $\mathcal{H}_\varphi$ so as to obtain completeness. This is achieved by introducing the Sobolev space $H_1(D)$, consisting of functions v on D, which have *weak derivatives* ∇v in D, and such that ∇v is square integrable on D. One should make sense of the boundary condition $v|\partial D = \varphi$. Now, minimize E over the appropriate subspace of admissible functions v, corresponding to the boundary conditions. This produces a minimizer for E in $H_1(D)$. The final task is to show regularity, i.e. that the solution is actually smooth, and the boundary conditions are assumed continuously. Here one uses the differential equation, Equation (1.12).

A variant of Theorem 1.1 will be proved in Chapter 3, see Theorem 3.5. Here, we will only prove the very last statement. Suppose $u \in C^1(D) \cap C^0(\overline{D})$, and minimizes E over $\mathcal{H}_\varphi$, then for any $w \in C_0^1(D)$, and any $t \in \mathbb{R}$, we have $u + tw \in \mathcal{H}_\varphi$, hence

$$\frac{d}{dt}\, E(u+tw)|_{t=0} = 0,$$

or,

$$\int_D \nabla u \cdot \nabla w = 0.$$

This is the *weak form* of (1.12). Suppose in addition that $u \in C^2(D)$, then integrating by parts, we find:

$$\int w\, \Delta u = 0,$$

for any $w \in C_0^1(D)$. By approximation, this holds also for any $w \in C_0^0(D)$, see Exercise 1.4. Now, let $\epsilon > 0$, and let $\chi \in C^\infty(D)$ be a cut-off function: $\chi = 1$ in $D_{1-\epsilon}$, $\chi = 0$ in $D_{1-\epsilon/2}$, and $0 \leq \chi \leq 1$. Then, take $w = \chi\, \Delta u$, to get $\Delta u = 0$ in $D_{1-\epsilon}$. Since $\epsilon > 0$ is arbitrary, (1.12) follow.

Once Dirichlet's principle is proved, it remains to minimize over all continuous monotonic parameterizations of the boundary. This is where the three points condition is used. In these notes, we will take the approach of [16], and minimize E directly over all monotonic parameterizations of the boundary which satisfy the three points condition.

To finish the solution of Problem 1.1, one prove the absence of branch points, i.e. points where ∇X is degenerate. This question was answered for $n = 3$ by Osserman [11]. An area minimizing surface cannot have interior branch points. The proof uses the Weierstrass representation in an essential way. For $n \geq 4$, the result is not true: area minimizing surfaces may have branch points.

1.3 Related Questions

Finally, we indicate possible further directions. First, one may wish to generalize this result to Riemannian manifolds. An n-dimensional Riemannian manifold M is a subset of $\mathbb{R}^N$ which is locally defined as the zero set of a C^∞ non-singular function into $\mathbb{R}^{N-n}$. That is, for every point $p \in M$, there is a ball $B \subset \mathbb{R}^N$, and a smooth function $F\colon B \to \mathbb{R}^{N-n}$ such that ∇F is non-singular in B, and $M \cap B = \{F = 0\}$. The inverse mapping theorem then allows one to construct n local coordinates near each point $p \in M$.

Also, for each point $p \in M$, the tangent space is an n-dimensional affine subspace of $\mathbb{R}^N$, which is made into a vector space by choosing p as the origin, and on which one induces a canonical inner product g, by using the Euclidean inner product on $\mathbb{R}^N$. Given a local system of coordinates on M, it is easily checked that the tangent to the coordinate curves form a basis for the tangent space, in terms of which the inner product is represented as a smooth symmetric $n \times n$-matrix valued function. Thus, locally we have the situation described earlier: a domain $\Omega \subset \mathbb{R}^n$ equipped with a Riemannian metric g_{ij}.

Now the problem is posed in identical fashion. Given a simple Jordan curve Γ in M, which bounds a disk type surface of finite area contained in M, is there a disk type surface S contained in M whose boundary is Γ, and which is minimal among all such surfaces? This question was answered in the affirmative by Morrey, who introduced for this purpose the notion of *harmonic maps* [9, 8].

Next one may wish to minimize over other parameter domains. For example, one may wish to have more than one boundary component. With two boundary components, for example, the parameter domain would have to be doubly connected. However, not all such domains are conformally equivalent [3].

In some cases, if one is seeking the surface of absolute minimum area, the solution may not be orientable. For some boundary curves, the solution is not even an immersed surface. In order to find the absolute minimum, one may need to enlarge the class of admissible surfaces. This leads into the subject of geometric measure theory which is not within the scope of this course [5, 15, 2].

Yet another completely different direction is to consider the *non-parametric* problem. Suppose the surface $S \subset \mathbb{R}^3$ is to be described as the graph of a function u over a domain $\Omega \subset \mathbb{R}^2$. Then, the area of S is given by:

$$A(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}. \tag{1.13}$$

If the surface S is of least area, then u must satisfy the elliptic nonlinear partial differential equation:

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0,$$

or, taking (x, y) as the coordinates in Ω:

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0. \tag{1.14}$$

This is the *non-parametric minimal surface* equation. One of the earliest theorems on minimal surfaces regards entire solutions of Equation (1.14): the Bernstein Theorem [10].

Theorem 1.2 BERNSTEIN THEOREM
If u is a C^2 solution of (1.14) *over the entire plane, then u is linear.*

One may regard this theorem as a non-linear version of Liouville's theorem. Notice however the absence of any growth condition.

1.4 Exercises

Exercise 1.1 Let $X \in C^1(D) \cap C^0(\overline{D})$ be a solution of Problem 1.1. Show that for any $Y \in C_0^1(D)$, there holds

$$\int_D \left\{\frac{(fX_u - gX_v)\cdot Y_u + (eX_v - gX_u)\cdot Y_v}{\sqrt{ef - g^2}}\right\} = 0. \tag{1.15}$$

This is the *weak form* of (1.2). Suppose in addition that $X \in C^2(D)$, then show that X satisfies (1.2).

Exercise 1.2 Let $\gamma \in C^1([0,1], \Omega)$ join the points p and $q \in \Omega$, i.e. $\gamma(0) = p$, $\gamma(1) = q$. Suppose γ minimizes E among all curves joining p and q within Ω. Show that for any $\tau \in C^1([0,1], \mathbb{R}^n)$, satisfying $\tau(0) = \tau(1) = 0$, there holds:

$$\int_0^1 \left(g(\dot\gamma, \dot\tau) + \frac{1}{2}\sum_{i,j,k} \frac{\partial g_{ij}}{\partial x^k}\dot\gamma^i\dot\gamma^j\tau^k\right) = 0.$$

Suppose in addition that $\gamma \in C^2([0,1],\Omega)$, then show that γ satisfies the *geodesic equation*, i.e. for $j = 1, \ldots, n$:

$$\ddot{\gamma}^j + \sum_{k,l} \Gamma^j_{kl} \dot{\gamma}^k \dot{\gamma}^l = 0, \tag{1.16}$$

where Γ^j_{kl} are the *Christoffel symbols* of the metric g. These are defined by:

$$\Gamma^j_{kl} = \frac{1}{2} g^{jm} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right). \tag{1.17}$$

where g^{jm} is the inverse of g_{jm}, i.e. $g^{jm} g_{mk} = \delta^j_k$.

Exercise 1.3 Let $X \in C^1(D, \mathbb{R}^n)$. Show that $E(X)$ is invariant under conformal transformations, i.e. if $\psi \colon D' \to D$ is conformal, then:

$$E(X \circ \psi) = E(X).$$

Exercise 1.4 Let $v \in C^0(D)$, and suppose

$$\int_D wv = 0, \tag{1.18}$$

for all $w \in C^1_0(D)$. Show that $v = 0$.

Exercise 1.5 Let A be defined as in (1.13) Let $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$, and suppose that $A(u) \leq A(v)$ for all $v \in C^1(\Omega) \cap C^0(\overline{\Omega})$ with $v|\partial\Omega = u|\partial\Omega$. Then, show that for any $w \in C^1_0(\Omega)$, there holds:

$$\int_\Omega \frac{\nabla u \cdot \nabla w}{\sqrt{1 + |\nabla u|^2}} = 0.$$

Suppose in addition that $u \in C^2(\Omega)$, then show that (1.14) holds.

Chapter 2

Analytic Preliminaries

In this chapter, we bring a few results from Sobolev spaces and elliptic regularity theory. Some proofs are included, but mostly we refer to the bibliography [14, 6, 1]. Throughout $\Omega \subset \mathbb{R}^n$ is bounded, and $\Omega' \subset\subset \Omega$ is taken to mean that Ω' is open with compact closure in Ω. All vector spaces are over the reals.

2.1 Functional Analysis

Recall that a *Banach space* $\mathfrak{B}$ is a vector space with a norm $\|\cdot\| : \mathfrak{B} \to [0, \infty)$, such that $\mathfrak{B}$ is complete with respect to $\|\cdot\|$. A *Hilbert space* $\mathfrak{H}$ is a Banach space in which the norm is derived from an inner product $(\,\cdot\,,\,\cdot\,)$:

$$\|x\|^2 = (x, x).$$

The most common examples are the $L^p(\Omega)$ spaces:

$$\begin{cases} L^p(\Omega) = \{f\colon \Omega \to \mathbb{R};\ \int_\Omega |f|^p < \infty\}, \\ \|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p\right)^{1/p} \end{cases}$$

Note that $\|\cdot\|_{L^p(\Omega)}$ is not a norm, since from $\|f\|_{L^p(\Omega)} = 0$, one can only conclude that $f = 0$ a.e. in Ω. Thus, one must consider equivalence classes of functions, where two functions are equivalent if they are equal except on a set of measure zero. When $p = \infty$, $L^\infty(\Omega)$ is defined to be the set of all essentially bounded functions on Ω and $\|f\|_{L^\infty(\Omega)}$ is the essential supremum of f over Ω. When $p = 2$, we have a Hilbert space, with the inner product:

$$(f, g) = \int_\Omega fg.$$

Other important Banach spaces in partial differential equations are the Hölder spaces $C^{k,\alpha}(\Omega)$ consisting of functions $f \in C^k(\Omega)$ whose derivatives of order k are Hölder continuous with exponent α. The norm on $C^{k,\alpha}(\Omega)$ is given by:

$$\|f\|_{C^{k,\alpha}(\Omega)} = \sum_{j=1}^{k} \sup_{\Omega} \left|\nabla^j f\right| + \sup_{x,y\in\Omega} \frac{\left|\nabla^k f(x) - \nabla^k f(y)\right|}{|x-y|^{\alpha}}.$$

The space $C^{0,\alpha}(\Omega)$ is denoted by $C^{\alpha}(\Omega)$.

Recall that a sequence $x_n \in \mathfrak{H}$, a Hilbert space, *converges weakly* to $x \in \mathfrak{H}$, written $x_n \rightharpoonup x$, if $(x_n, y) \to (x, y)$ for every $y \in \mathfrak{H}$. The unit ball is not compact in an infinite dimensional Hilbert space, however one has the following important theorem [14].

Theorem 2.1 *Let $\mathfrak{H}$ be a Hilbert space, then the closed unit ball in $\mathfrak{H}$ is weakly compact. Consequently, every bounded sequence in $\mathfrak{H}$ has a weakly convergent subsequence.*

2.2 Sobolev Spaces

Definition 2.1 *Let $f \in L^1_{\mathrm{loc}}(\Omega)$, then we say that the vector field $X \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ is the* weak gradient *of f if*

$$\int_\Omega f \operatorname{div} Y = -\int_\Omega X \cdot Y, \tag{2.1}$$

for every vector field $Y \in C_0^\infty(\Omega;\mathbb{R}^n)$.

In particular, if $\varphi \in C_0^\infty(\Omega)$, and $\vec{e}_k$ is the standard orthonormal basis in $\mathbb{R}^n$, then we have:

$$\int_\Omega f \, \nabla_{\vec{e}_k} \varphi = -\int_\Omega X \cdot \vec{e}_k \, \varphi.$$

To justify this definition, we prove the following elementary theorem:

Proposition 2.1 *The weak gradient is unique, i.e. if X and X' are both weak gradients of f then $X = X'$ a.e. in Ω.*

To prove this theorem, we need a density lemma. For a measurable function v on Ω, let $\operatorname{supp} v$ be the smallest closed set K such that $v = 0$ a.e. in $\Omega \setminus K$. Define $\mathring{L}^\infty(\Omega)$ to be the subset of $L^\infty(\Omega)$ consisting of those functions v for which $\operatorname{supp} v$ is compact in Ω. Also for $\varepsilon > 0$ let $\Omega_\varepsilon = \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) > \varepsilon)\}$, and let χ_ε be the characteristic function of Ω_ε.

Lemma 2.1 *Let* $u \in L^1_{\text{loc}}(\Omega)$ *satisfy*

$$\int_\Omega uv = 0, \qquad \forall v \in C_0^\infty(\Omega), \tag{2.2}$$

then (2.2) *holds also* $\forall v \in \overset{\circ}{L}{}^\infty(\Omega)$. *If* $u \in L^1(\Omega)$, *then* (2.2) *holds* $\forall v \in L^\infty(\Omega)$.

In the language of functional analysis, this shows that $C_0^\infty(\Omega)$ is weak-$*$ dense in $L^\infty(\Omega)$.

Proof of Lemma 2.1. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be defined by

$$\varphi(x) = \begin{cases} b \exp\left(\dfrac{1}{|x|^2 - 1}\right) & \text{for } |x| \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

with b chosen so that $\int \varphi = 1$. For $\varepsilon > 0$, define $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Then, it is easy to check that for every $\varepsilon > 0$

$$\int \varphi_\varepsilon = 1,$$

$0 \le \varphi_\varepsilon \le \varepsilon^{-n} b$, and φ_ε vanishes outside $B_\varepsilon(0)$. The functions φ_ε are called *mollifiers*. Consider first a function $v \in \overset{\circ}{L}{}^\infty(\Omega)$. Let $\delta = \operatorname{dist}(\operatorname{supp} v, \partial\Omega)$, then for $0 < \varepsilon < \delta/2$, the function

$$v_\varepsilon(x) = \int_\Omega v(y)\, \varphi_\varepsilon(x - y)\, dy$$

belongs to $C_0^\infty(\Omega)$, hence by (2.2):

$$\int_\Omega u\, v_\varepsilon = 0. \tag{2.3}$$

Now

$$|v(x) - v_\varepsilon(x)| = \left| \int_\Omega \Big(v(x) - v(y)\Big)\varphi_\varepsilon(x - y\Big)\, dy \right| \le \frac{b}{\varepsilon^n} \int_{B_\varepsilon(x)} |v(x) - v(y)|\,,$$

which tends to zero as $\varepsilon \to 0$, for every Lebesgue point of v, and hence a.e. in Ω. Furthermore,

$$|v_\varepsilon(x)| \le \left| \int_\Omega v(y)\, \varphi_\varepsilon(x - y) dy \right| \le \|v\|_{L^\infty(\Omega)}\,,$$

and hence, for $0 < \varepsilon < \delta/2$, we have pointwise:

$$|u\, v_\varepsilon| \le \|v\|_{L^\infty(\Omega)}\, |u|\, \chi_{\delta/2} \in L^1(\Omega).$$

It follows, using (2.3) and the dominated convergence theorem, that (2.2) holds for v. This proves the first statement in the lemma. To finish the proof, let $v \in L^\infty(\Omega)$. Then multiply v by χ_ε to get a function $\chi_e v \in \overset{\circ}{L}{}^\infty(\Omega)$. Clearly, $\chi_\varepsilon v \to v$ pointwise as $\varepsilon \to 0$, hence using the previous result together with the dominated convergence theorem again, one obtains the second statement in the lemma. $\square$

Proof of Proposition 2.1. Since both X and X' are weak gradients of f, we have

$$\int_\Omega (X - X') \cdot Y = 0,$$

for every $Y \in C_0^\infty(\Omega; \mathbb{R}^n)$. By the lemma, the same holds for all $Y \in \overset{\circ}{L}{}^\infty(\Omega; \mathbb{R}^n)$. Let $\varepsilon > 0$, and take

$$Y = \begin{cases} \chi_\varepsilon (X - X')/\left|X - X'\right| & \text{if } X \neq X', \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly $Y \in \overset{\circ}{L}{}^\infty(\Omega; \mathbb{R}^n)$, hence

$$\int_\Omega (X - X') \cdot Y = \int_{\Omega_\varepsilon} |X - X'| = 0,$$

and therefore $X = X'$ a.e. in Ω_ε. It follows that $X = X'$ a.e. in Ω. This completes the proof of the proposition. $\square$

Note that if $f \in C^1(\Omega)$, then by Green's Theorem, ∇f, the classical gradient of f, is also the weak gradient of f. We will write ∇f also for the weak gradient. Although the definition seems global, the weak gradient, just as the classical gradient, depends only locally on the values of f. More precisely, we have the following elementary lemma:

Lemma 2.2 *Let $f \in L^1_{\mathrm{loc}}(\Omega)$ have a weak gradient in Ω, and suppose $f = 0$ on an open set $U \subset \Omega$, then $\nabla f = 0$ a.e. in U.*

In fact, all the usual rules of calculus are preserved. We can now define our first Sobolev spaces.

Definition 2.2 *Let $1 \leq p \leq \infty$. The spaces $W^{1,p}(\Omega)$ consists of those functions $f \in L^p(\Omega)$ whose weak gradient ∇f belongs to $L^p(\Omega; \mathbb{R}^n)$. The norm on $W^{1,p}(\Omega)$ is defined by:*

$$\|f\|^p_{W^{1,p}(\Omega)} = \|f\|^p_{L^p(\Omega)} + \|\nabla f\|^p_{L^p(\Omega;\mathbb{R}^n)}.$$

An inner product on $H_1(\Omega) = W^{1,2}(\Omega)$ is defined by:

$$(f, g)_{H_1(\Omega)} = \int_\Omega \left(fg + \nabla f \cdot \nabla g\right).$$

Theorem 2.2 *The spaces $W^{1,p}(\Omega)$ are Banach spaces. The space $H_1(\Omega)$ is a Hilbert space.*

The proof of this theorem is left to the exercises.

Higher order derivatives are defined similarly. We will only present briefly the second order derivatives. We say that $f \in W^{1,1}_{\text{loc}}(\Omega)$ if $f \in W^{1,1}(\Omega')$ for every $\Omega' \subset\subset \Omega$. Let $S^{n\times n}$ denote the vector space of symmetric linear transformations $A\colon \mathbb{R}^n \to \mathbb{R}^n$. If $A \in C^1(\Omega; S^{n\times n})$, the divergence of A is the vector field defined by

$$(\operatorname{div} A) = \sum_k \nabla_{\vec{e}_k} A\vec{e}_k,$$

where $\vec{e}_k$ is an orthonormal basis for $\mathbb{R}^n$, and $\nabla_{\vec{e}_k} = \vec{e}_k \cdot \nabla$. An inner product in $S^{n\times n}$ is defined by

$$A \cdot B = \sum_k A\vec{e}_k \cdot B\vec{e}_k.$$

Definition 2.3 *Let $f \in W^{1,1}_{\text{loc}}(\Omega)$, then we say that $A \in L^1_{\text{loc}}(\Omega; S^{n\times n})$ is the* weak hessian *of f if*

$$\int_\Omega \nabla f \cdot \operatorname{div} B = -\int_\Omega A \cdot B, \tag{2.4}$$

for all $B \in C_0^\infty(\Omega; S^{n\times n})$.

In particular, if $\varphi \in C_0^\infty(\Omega)$, then

$$\int_\Omega f \nabla_{\vec{e}_k} \nabla_{\vec{e}_l} \varphi = \int_\Omega A\vec{e}_k \cdot \vec{e}_l\, \varphi.$$

We leave it as an exercise to check that this is well defined, see Exercise 2.2. Note that although the hessian is not necessarily continuous, the mixed weak partial derivatives do not depend on the order of differentiation. Thus, in this respect, weak derivatives are better behaved than the classical partial derivatives. Furthermore, if $f \in C^2(\Omega)$, then $\nabla^2 f$, the classical hessian, is also the weak hessian. This follows from Stokes' Theorem. We will write $\nabla^2 f$ also for the weak hessian. The space $W^{2,p}(\Omega)$ can now be defined.

Definition 2.4 *Let $1 \leq p \leq \infty$. The space $W^{2,p}(\Omega)$ consists of those functions $f \in W^{1,p}(\Omega)$ whose weak hessian $\nabla^2 f$ belongs to $L^p(\Omega; S^{n\times n})$. The norm on $W^{2,p}(\Omega)$ is defined by:*

$$\|f\|^p_{W^{2,p}(\Omega)} = \|f\|^p_{W^{1,p}(\Omega)} + \left\|\nabla^2 f\right\|^p_{L^p(\Omega; S^{n\times n})}$$

An inner product on the space $H_2(\Omega)$ is defined by:

$$(f,g)_{H_2(\Omega)} = (f,g)_{H_1(\Omega)} + \int_\Omega \nabla^2 f \cdot \nabla^2 g.$$

It is now easy to generalize and define the spaces $W^{k,p}(\Omega)$, for any integer $k \geq 0$. As before, $H_k(\Omega)$ is used to denote $W^{k,2}(\Omega)$.

Theorem 2.3 *For each $1 \leq p \leq \infty$, and each non-negative integer k, $W^{k,p}(\Omega)$ is a Banach spaces. $H_k(\Omega)$ is a Hilbert space.*

The following theorem is found to be very useful. It allows one to reduce many proofs to the case of smooth functions. Recall that a *locally finite cover of* Ω is a collection of open sets $\Omega_j \subset\subset \Omega$, such that $\cup_j \Omega_j = \Omega$, and for each point $p \in \Omega$, there is a neighborhood of p which intersects only finitely many of the Ω_j. A *partition of unity subordinate to the cover* Ω_j is a collection of functions $\varphi_j \in C^\infty(\Omega)$, such that $0 \leq \varphi_j \leq 1$, $\varphi_j \in C_0^\infty(\Omega_j)$, and $\sum_j \varphi_j = 1$ everywhere in Ω.

Theorem 2.4 *The subspace $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.*

Proof. The proof requires a lemma similar to Lemma 2.1. Define S_ε to be the smoothing operator:

$$S_\varepsilon v(x) = \int_\Omega v(y)\, \varphi_\varepsilon(x-y) dy. \tag{2.5}$$

Lemma 2.3 *Let $v \in L^p(\Omega)$, then $S_\varepsilon v \to v$ in $L^p(\Omega)$.*

We leave out the proof of this lemma [6], and return to the proof of the theorem. Let $\mathring{W}^{k,p}(\Omega)$ be the subspace of $W^{k,p}(\Omega)$ consisting of those functions having compact support in Ω, and let $\mathring{L}^p(\Omega) = \mathring{W}^{0,p}(\Omega)$. First we see that $C_0^\infty(\Omega)$ is dense in $\mathring{W}^{k,p}(\Omega)$. This follows from the lemma and the fact that $S_\varepsilon \nabla v = \nabla \Big(S_e v \Big)$ as long as $0 < 2\varepsilon <$ dist (supp $v, \partial\Omega$). Now, let Ω_j be a locally finite cover of Ω, and let φ_j be a partition of unity subordinate to the cover Ω_j. Let $u \in W^{k,p}(\Omega)$, and $\varepsilon > 0$. Then for each j, $\varphi_j u \in \mathring{W}^{k,p}(\Omega_j)$, hence applying the previous result, there is a function $v_j \in C_0^\infty(\Omega_j)$ such that

$$\|\varphi_j u - v_j\|_{W^{k,p}(\Omega_j)} \leq 2^{-j}\varepsilon.$$

Extend v_j to be zero outside Ω_j and let $v = \sum_j v_j$. Since the cover Ω_j is locally finite, for every $\Omega' \subset\subset \Omega$, there are only finitely v_j which are not identically zero on Ω'. Thus, $v \in C^\infty(\Omega)$. Furthermore,

$$\|u - v\|_{W^{k,p}(\Omega)} \leq \sum_j \|\varphi_j u - v_j\|_{W^{k,p}(\Omega_j)} \leq \varepsilon.$$

This competes the proof of the theorem. □

This motivates the following definition.

Definition 2.5 *The space $W_0^{k,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. We write $H_{k,0}(\Omega) = W_0^{k,2}(\Omega)$.*

As the following proposition shows [1], this definition allows the generalization of Dirichlet boundary conditions to $W^{1,p}(\Omega)$.

Proposition 2.2 *Let $f \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$, then $f|\partial\Omega = 0$ if and only if $f \in W_0^{1,p}(\Omega)$.*

2.3 Sobolev Inequalities

Often, solutions of elliptic variational problems are obtained as weak solutions, i.e. as elements of some Sobolev space, $u \in W^{k,p}(\Omega)$. Regularity is then usually obtained by showing that u has weak derivatives of high enough order which are integrable to some power. The results of this section then show that u has continuous classical derivatives of a lower order. There are two types of results. The first yields higher integrability, while the second yields continuity. Throughout, we will assume that $\Omega \subset \mathbb{R}^n$ is bounded, and has a smooth boundary. A Lipschitz boundary would be enough, but we will apply these mostly to the disk in $\mathbb{R}^2$.

Theorem 2.5 *Let $u \in W^{1,1}(\Omega)$, then $u \in L^q(\Omega)$ for $1 \leq q \leq n/(n-1)$. Furthermore there exists a constant C, depending only on n, Ω, and q, such that*

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{1,1}(\Omega)}, \tag{2.6}$$

for every $u \in W^{1,1}(\Omega)$.

We will not present a proof of this theorem [6, 1]. It suffices to say that, thanks to Theorem 2.4, it is only necessary to show that Inequality (2.6) holds for all functions $u \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$. From this theorem it is not difficult to obtain the following.

Theorem 2.6 *Let $p < n$, and $u \in W^{1,p}(\Omega)$, then $u \in L^q(\Omega)$ for $1 \leq q \leq np/(n-p)$. Furthermore, there is a constant C depending only on n, Ω, p, and q, such that*

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}, \tag{2.7}$$

for every $u \in W^{1,p}(\Omega)$.

Proof. It suffices to check the critical case $q = np/(n-p)$. We will need the following lemma whose proof is left to the exercises.

Lemma 2.4 *Let $1 \le p \le r \le q$, then $L^p(\Omega) \cap L^q(\Omega) \hookrightarrow L^r(\Omega)$, and furthermore, for $\forall u \in L^p(\Omega) \cap L^q(\Omega)$, we have*

$$\|u\|_{L^r(\Omega)} \le \|u\|^{\lambda}_{L^p(\Omega)} \|u\|^{1-\lambda}_{L^q(\Omega)}, \tag{2.8}$$

where λ is given by $1/r = \lambda/p + (1-\lambda)/q$.

Now let $u \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$, and define

$$u_M = \begin{cases} -M & \text{where } u < -M, \\ u & \text{where } |u| \le M, \\ M & \text{where } u > M. \end{cases}$$

Then, one checks that $u_M \in W^{1,p}(\Omega)$ and the weak gradient of u_M is

$$\nabla u_M = \chi_M \nabla u,$$

where χ_M is the characteristic function of the set $\{x \in \Omega;\ |u(x)| < M\}$. Let $v_M = |u_M|^r$, where $r = p(n-1)/(n-p)$, then calculate

$$|\nabla v_M| = r\chi_M |u|^{r-1} |\nabla u|.$$

It follows that $v_M \in W^{1,1}(\Omega)$. In fact, using (2.8), we have

$$\int_\Omega |v_M| \le \left(\int_\Omega |u|^p\right)^{\lambda r/p} \left(\int_\Omega |u_M|^q\right)^{(1-\lambda)r/q}, \tag{2.9}$$

where $1/r = \lambda/p + (1-\lambda)/q$. Also, in view of $p(r-1)/(p-1) = q$, we find

$$\int_\Omega |\nabla v_M| \le r\left(\int_\Omega |u_M|^q\right)^{1-1/p} \left(\int_\Omega |\nabla u|^p\right)^{1/p}. \tag{2.10}$$

Noting that $\lambda r = 1$, and $(1-\lambda)r = q(1-1/p)$, these inequalities can be written as

$$\begin{aligned} \|v_M\|_{L^1(\Omega)} &\le \|u\|_{L^p(\Omega)} \|u_M\|^{q(1-1/p)}_{L^q(\Omega)}, \\ \|\nabla v_M\|_{L^1(\Omega;\mathbb{R}^n)} &\le r\|\nabla u\|_{L^p(\Omega;\mathbb{R}^n)} \|u_M\|^{q(1-1/p)}_{L^q(\Omega)}, \end{aligned}$$

from which, we obtain:

$$\|v_M\|_{W^{1,1}(\Omega)} \le r\|u\|_{W^{1,p}(\Omega)} \|u_M\|^{q(1-1/p)}_{L^q(\Omega)}.$$

Now, using Inequality (2.6), we have

$$\|u_M\|^{q(1-1/n)}_{L^q(\Omega)} = \|v_M\|_{L^{n/(n-1)}(\Omega)} \le C\|v_M\|_{W^{1,1}(\Omega)} \le Cr\|u\|_{W^{1,p}(\Omega)} \|u_M\|^{q(1-1/p)}_{L^q(\Omega)},$$

which, upon cancellation of $\|u_M\|^{q(1-1/p)}_{L^q(\Omega)}$, yields

$$\|u_M\|_{L^q(\Omega)} \le Cr\|u\|_{W^{1,p}(\Omega)}.$$

By the monotone convergence theorem, we may now let $M \to \infty$, and we conclude that Inequality (2.7) holds for all $u \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$. The rest follows by the density result, Theorem 2.4. □

In the language of functional analysis, this theorem says that $W^{1,p}(\Omega)$ is naturally embedded in $L^q(\Omega)$ for $1 \leq q \leq np/(n-p)$, and that the embedding is continuous. In fact, a stronger version is true [6, 1]. Recall that a linear map between Banach spaces is *compact* if it sends bounded sets to pre-compact ones.

Theorem 2.7 *The embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact if $q < np/(n-p)$.*

Thus for instance the embedding of $H_1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. This is usually referred to as Rellich's Theorem. One of the useful consequences of this theorem is that a sequence which converges weakly in $H_1(\Omega)$ converges strongly in $L^2(\Omega)$.

Enough integrability of the derivatives implies the continuity of the function [6, 1].

Theorem 2.8 *Let $p > n$, and $u \in W^{1,p}(\Omega)$, then $u \in C^\alpha(\Omega)$, where $\alpha = 1 - n/p$. Furthermore, there exists a constant C depending only on n, Ω, and p, such that*

$$\|u\|_{C^\alpha(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}, \tag{2.11}$$

for all $u \in W^{1,p}(\Omega)$.

Note that the case $p = n$ is not included. Thus functions in $H_1(\Omega)$ are not necessarily continuous, even when $n = 2$. A typical application of these results is given in the next section.

2.4 Elliptic Regularity

Definition 2.6 *A function $u \in C^2(\Omega)$ is called* harmonic *if it satisfies the equation:*

$$\Delta u = 0. \tag{2.12}$$

Let $\varphi \in C_0^\infty(\Omega)$, multiply (2.12) by φ, and integrate by parts to get:

$$\int_\Omega \nabla u \cdot \nabla \varphi = 0. \tag{2.13}$$

This integral is still well defined for $u \in H_1(\Omega)$. Thus we define:

Definition 2.7 *Let $u \in H_1(\Omega)$ satisfy (2.13) for all $\varphi \in C_0^\infty(\Omega)$, then we say that u is* weakly harmonic.

The goal of this section is to study the interior and boundary regularity of weakly harmonic functions. We begin with a few classical result. The first is known as the *mean value theorem*.

Theorem 2.9 *Let u be harmonic on Ω. Then for every ball $B_r(x) \subset\subset \Omega$, we have*

$$u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy, \tag{2.14}$$

where $|B_r(x)|$ is the volume of $B_r(x)$.

A consequence this theorem is the *maximum principle.*

Theorem 2.10 *Let u be harmonic on Ω, and suppose that there exists a point $x \in \Omega$ for which $u(x) = \sup_\Omega u$. Then u is constant. Thus, for every harmonic $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$:*

$$\inf_{\partial\Omega} u \leq u(x) \leq \sup_{\partial\Omega} u,$$

for all $x \in \Omega$.

This in turn implies a uniqueness result.

Theorem 2.11 *Let $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy $\Delta u = \Delta v$ in Ω, and $u = v$ on $\partial\Omega$. Then $u = v$ in Ω.*

Recall that a function $u \in C^\infty(\Omega)$ is *real analytic*, $u \in C^\omega(\Omega)$, if for every $\Omega' \subset\subset \Omega$, the Taylor series of u converges to u uniformly on Ω'.

Theorem 2.12 *Every harmonic function u on Ω is real analytic.*

Sketch of proof. This is usually done in two steps. First, using Theorem 2.9, one shows that $u \in C^\infty(\Omega)$. To show analyticity one needs to control the growth of the derivatives. This is the content of Harnack's inequality:

Proposition 2.3 *Let $u \in C^2(\Omega)$ be harmonic, $\Omega' \subset\subset \Omega$, and $d = \operatorname{dist}(\Omega', \partial\Omega)$. Then*

$$\sup_{\Omega'} |\nabla u| \leq \left(\frac{n}{d}\right) \sup_{\Omega} |u|. \tag{2.15}$$

Since the partial derivatives of u are also harmonic, this gives, by induction, control of all the derivatives of u and allows one to show analyticity. □

We now turn to weak solutions of $\Delta u = 0$.

Proposition 2.4 *Let $u \in L^2(\Omega)$ satisfy*

$$\int_\Omega u \Delta\varphi = 0, \tag{2.16}$$

for all $\varphi \in C_0^\infty(\Omega)$. Then, for each $\Omega' \subset\subset \Omega$, u belongs to $H_1(\Omega')$. Furthermore, there exists a constant C depending only on n, and on $d = \operatorname{dist}(\Omega', \partial\Omega)$, such that

$$\|\nabla u\|_{L^2(\Omega';\mathbb{R}^n)} \leq C \|u\|_{L^2(\Omega)}. \tag{2.17}$$

Proof. Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, and let $2\varepsilon < \operatorname{dist}(\Omega'', \partial\Omega)$. Let $u_\varepsilon = S_\varepsilon$, where S_ε is defined by (2.5), then $u \in C^\infty(\Omega'')$, and for $x \in \Omega''$, we have

$$\begin{aligned} \Delta u_\varepsilon &= \int_\Omega u(y)\, \Delta_x \varphi_\varepsilon(x-y)\, dy \\ &= \int_\Omega u(y)\, \Delta_y \varphi_\varepsilon(x-y)\, dy \\ &= 0, \end{aligned}$$

by (2.16). Here Δ_x and Δ_y denote the Laplacian with respect to the x and y variables respectively. Thus, u_ε is harmonic in Ω''. Let $\chi \in C_0^\infty(\Omega'')$ be a cut-off function for Ω', satisfying:

$$\begin{cases} \chi = 1 & \text{in } \Omega', \\ |\nabla\chi| \le c/d & \text{in } \Omega''. \end{cases}$$

Multiplying the equation $\Delta u_\varepsilon = 0$ by χu_ε, and integrating by parts, we obtain

$$\begin{aligned} 0 &= \int_{\Omega''} \chi^2 u_\varepsilon \Delta u_\varepsilon \\ &= -\int_{\Omega''} \nabla(\chi^2 u_\varepsilon)\cdot\nabla u_\varepsilon \\ &= -\int_{\Omega''} \chi^2 \left|\nabla u_\varepsilon\right|^2 - 2\int_{\Omega''} \chi u_\varepsilon \nabla\chi\cdot\nabla u_\varepsilon \\ &\le -\int_{\Omega''} \chi^2 \left|\nabla u_\varepsilon\right|^2 + \sup_{\Omega''}|\nabla\chi| \left(\int_{\Omega''} u_\varepsilon^2\right)^{1/2} \left(\int_{\Omega''} \chi^2 \left|\nabla u_\varepsilon\right|^2\right)^{1/2}. \end{aligned}$$

Furthermore, for $x \in \Omega''$, we have:

$$|S_\varepsilon u(x)|^2 \le \int_{\Omega''} u^2(y)\, \varphi_\varepsilon(x-y) dy,$$

hence,

$$\|u_\varepsilon\|^2_{L^2(\Omega'')} \le \int_{\Omega''}\int_{\Omega''} u^2(y)\, \varphi_\varepsilon(x-y)\, dy\, dx = \|u\|^2_{L^2(\Omega'')}\,.$$

It follows that

$$\|\nabla u_\varepsilon\|_{L^2(\Omega';\mathbb{R}^n)} \le \left(\frac{c}{d}\right)\|u\|_{L^2(\Omega)}\,. \tag{2.18}$$

Therefore, u_ε is uniformly bounded in $H_1(\Omega')$, hence there is a sequence $\varepsilon_j \to 0$ such that $u_{\varepsilon_j} \rightharpoonup v$ in $H_1(\Omega')$. By the compactness of the embedding $H_1(\Omega') \hookrightarrow L^2(\Omega')$, we have $u_{\varepsilon_j} \to v$ strongly in $L^2(\Omega)$. But by Lemma 2.3, we also have $u_{\varepsilon_j} \to u$ in $L^2(\Omega')$, hence $v = u$, i.e. $u \in H_1(\Omega')$, and $u_{\varepsilon_j} \rightharpoonup u$ in $H_1(\Omega')$. Now, we can estimate:

$$\begin{aligned}
\int_{\Omega'} |\nabla u|^2 &= \lim_j \int_{\Omega'} \nabla u \cdot \nabla u_{\varepsilon_j} \\
&\leq \left(\int_{\Omega'} |\nabla u|^2\right)^{1/2} \liminf_j \left(\int_{\Omega'} \left|\nabla u_{\varepsilon_j}\right|^2\right)^{1/2} \\
&\leq \left(\frac{c}{d}\right) \|u\|_{L^2(\Omega)} \left(\int_{\Omega'} |\nabla u|^2\right)^{1/2},
\end{aligned}$$

from which (2.17) follows. This completes the proof of the proposition. $\square$

We can now prove the following interior regularity theorem.

Theorem 2.13 *Let $u \in H_1(\Omega)$ be weakly harmonic. Then $u \in C^\omega(\Omega)$.*

Proof. For simplicity, we will only consider the case $n = 2, 3$. We will prove that, for each $\Omega' \subset\subset \Omega$, $u \in H_4(\Omega')$. Then, by Theorem 2.6, $\nabla^3 u \in W^{1,6}(\Omega')$, and hence by Theorem 2.8, $\nabla^2 u \in C^\alpha(\Omega')$. It follows that $u \in C^2(\Omega')$, hence by Theorem 2.12, u is analytic in Ω'. To show $u \in H_4(\Omega')$, we proceed by induction. We will only present the first step; the rest is similar. Let $w = \nabla_{\vec{v}} u$, where $\vec{v}$ is a unit vector in $\mathbb{R}^n$. We have $w \in L^2(\Omega)$, and we must show that $w \in H_1(\Omega')$. Let $\varphi \in C_0^\infty(\Omega)$, and let $\psi = \nabla_{\vec{v}}\varphi$. Clearly $\psi \in C_0^\infty(\Omega)$, hence

$$\begin{aligned}
\int_\Omega w\, \Delta\varphi &= -\int_\Omega u\, \nabla_{\vec{v}}\Delta\varphi \\
&= -\int_\Omega u\, \Delta\nabla_{\vec{v}}\varphi \\
&= \int_\Omega \nabla u \cdot \nabla\psi \\
&= 0.
\end{aligned}$$

Thus, by Theorem 2.4, it follows that $w \in H_1(\Omega')$, for every $\Omega' \subset\subset \Omega$. This completes the proof of the theorem. $\square$

It remains to discuss boundary regularity. We will only state the following result [6]. We say that a domain $\Omega \subset \mathbb{R}^n$, has boundary $\partial\Omega$ of class C^k, if it can locally be straightened out by a C^k map. More precisely, for each point $x \in \partial\Omega$, there is a C^k, one-to-one, non-singular map $F\colon B \to \mathbb{R}^n$, where $B \subset \mathbb{R}^n$ is the unit ball, such that $F(0) = x$, and $F^{-1}(\Omega) = \{x \in B;\ x^n > 0\}$.

Theorem 2.14 *Let $\partial\Omega$ be of class C^k, $k \geq 2$, and assume that $\varphi \in H_k(\Omega)$. Let u be (weakly) harmonic in Ω, and $u - \varphi \in H_{1,0}(\Omega)$. Then also $u \in H_k(\Omega)$. Furthermore, there exists a constant C, depending only on n, Ω, and k, such that*

$$\|u\|_{H_k(\Omega)} \leq C \left(\|u\|_{L^2(\Omega)} + \|\varphi\|_{H_k(\Omega)} \right).$$

Therefore, if $\partial\Omega$ is of class C^∞, and $\varphi \in C^\infty(\overline{\Omega})$, then also $u \in C^\infty(\overline{\Omega})$.

2.5 Exercises

Exercise 2.1 Show that $W^{1,p}(\Omega)$ is a Banach space. Show that for $1 < p < \infty$, the Banach space $W^{1,p}(\Omega)$ is reflexive.

Exercise 2.2 Let $f \in W^{1,1}_{\text{loc}}(\Omega)$, and let both A and A' be weak hessians of f. Show that $A = A'$, a.e. in Ω.

Exercise 2.3 Prove Lemma 2.4.

Exercise 2.4 Prove Theorem 2.9.

Exercise 2.5 Prove Theorem 2.10.

Exercise 2.6 Prove Theorem 2.11.

Chapter 3

Variational Principles

In this chapter, we prove a variant of Theorem 1.1 in $\mathbb{R}^n$, study related variational problems, and state a general variational principle. We assume that $\Omega \subset \mathbb{R}^n$ is bounded, and $\partial\Omega$ is of class C^∞. Define

$$E(u) = \int_\Omega |\nabla u|^2$$

for $u \in H_1(\Omega)$. We begin with the Poincaré inequality.

3.1 Poincaré Inequalities

Theorem 3.1 *There is a constant C depending only on Ω, such that for all $u \in H_{1,0}(\Omega)$,*

$$\int_\Omega u^2 \leq C \int_\Omega |\nabla u|^2 . \tag{3.1}$$

Proof. Let $B = \{u \in H_{1,0}(\Omega);\ \int_\Omega u^2 = 1\}$. We will minimize E over B. Let $c = \inf_B E$, and let $u_j \in B$ be a minimizing sequence, i.e. $E(u_j) \to c$. Then u_j is uniformly bounded in $H_{1,0}(\Omega)$, hence there exists a subsequence, which, without loss of generality, we call again u_j, such that $u_j \rightharpoonup u_0$ in $H_{1,0}(\Omega)$. Now, by the compactness of the embedding $H_1(\Omega) \hookrightarrow L^2(\Omega)$, $u_j \to u_0$ strongly in $L^2(\Omega)$, hence $u_0 \in B$. Furthermore, as in the proof of Proposition 2.4, E is weakly lower semi-continuous on B. Indeed,

$$\int_\Omega |\nabla u_0|^2 = \lim \int_\Omega \nabla u_0 \cdot \nabla u_j \leq \left(\int_\Omega |\nabla u_0|^2\right)^{1/2} \liminf \left(\int_\Omega |\nabla u_j|^2\right)^{1/2} .$$

Thus $c \leq E(u_0) \leq \lim E(u_j) = c$, and we conclude $E(u_0) = c$. In particular $c \neq 0$, for $c = 0$ would imply that $u_0 \equiv \text{const.}$, and since $u_0 \in H_{1,0}(\Omega)$, it would follow that

$u_0 = 0$, but this is impossible since $u_0 \in B$. Now let $u \in H_{1,0}(\Omega)$, and normalize u by:

$$u' = \frac{u}{\|u\|_{L^2(\Omega)}}.$$

Then $u' \in B$, hence $E(u') \geq E(u_0) = c$. This implies (3.1) with $C = c^{-1}$. $\square$

Corollary 3.1 *There is a constant C depending only on Ω, such that for all $\varphi, u \in H_1(\Omega)$, for which $u - \varphi \in H_{1,0}(\Omega)$,*

$$\int_\Omega u^2 \leq C \left(\int_\Omega |\nabla u|^2 + \|\varphi\|^2_{H_1(\Omega)} \right).$$

The proof is left to the reader. The second term is necessary, otherwise a contradiction is easily obtained by adding a large constant to u. Another way to prevent this possibility is to normalize by $\int_\Omega u = 0$.

Theorem 3.2 *There is a constant C depending only on Ω, such that for all $u \in H_1(\Omega)$ satisfying $\int_\Omega u = 0$, there holds*

$$\int_\Omega u^2 \leq C \int_\Omega |\nabla u|^2 \tag{3.2}$$

The proof is similar to that of Theorem 3.1 and is left as an exercise.

We end this section with the following result which will be needed in the next chapter. Although it can be stated and proved in any dimension, for simplicity we consider only the case $n = 2$. Let $D \subset \mathbb{R}^2$ be the unit disk.

Theorem 3.3 *For all $u \in H_1(D) \cap L^2(\partial D)$, the following inequality holds:*

$$\|u\|_{L^2(D)} \leq 2 \|\nabla u\|_{L^2(D;\mathbb{R}^2)} + \|u\|_{L^2(\partial D)}. \tag{3.3}$$

Remark. In fact, $H_1(D) \hookrightarrow L^2(\partial D)$, and this embedding is continuous. Thus, the condition $u \in L^2(\partial D)$ is redundant, but we will not discuss this point here.

Proof. It suffices to prove (3.3) for $u \in H_1(D) \cap C^\infty(D)$. For $0 < r < 1$ let $C_r = \partial D_r$ be the circle of radius r centered at the origin. For every $u \in C^\infty(D)$, define the average of u over C_r:

$$\overline{u}(r) = \frac{1}{2\pi} \int_{C_r} u \, d\theta. \tag{3.4}$$

By the Poincaré inequality on the unit circle, see Exercise 3.3, we have

$$\int_{C_r} \left(u - \overline{u}\right)^2 d\theta \leq \int_{C_r} u_\theta^2 \, d\theta, \tag{3.5}$$

hence

$$\begin{aligned}\|u-\overline{u}\|^2_{L^2(D)} &= \int_0^1\int_{C_r}\left(u-\overline{u}\right)^2 r\,d\theta\,dr\\ &\le \int_0^1\int_{C_r} u_\theta^2\, r\,d\theta\,dr\\ &\le \int_0^1\int_{C_r}\frac{1}{r}\,u_\theta^2\,d\theta\,dr\\ &= \left\|r^{-1}u_\theta\right\|^2_{L^2(D)}.\end{aligned}$$

It follows that

$$\|u\|_{L^2(D)} \le \left\|r^{-1}u_\theta\right\|_{L^2(D)} + \|\overline{u}\|_{L^2(D)}. \tag{3.6}$$

Now, since

$$\overline{u}^2 \le \frac{1}{2\pi}\int_{C_r} u^2\,d\theta,$$

we have

$$\begin{aligned}\|\overline{u}\|^2_{L^2(D)} &= 2\pi\int_0^1 \overline{u}^2\,r\,dr\\ &= \pi\,\overline{u}^2\,r^2\Big|_0^1 - 2\pi\int_0^1 \overline{u}\,\overline{u}_r\,r^2\,dr\\ &\le \pi\,\overline{u}^2(1) + 2\pi\left(\int_0^1\overline{u}^2\,r\,dr\right)^{1/2}\left(\int_0^1\overline{u}_r^2 r^3\,dr\right)^{1/2}\\ &\le \frac{1}{2}\,\|u\|^2_{L^2(\partial D)} + \|\overline{u}\|_{L^2(D)}\,\|r\overline{u}_r\|_{L^2(D)}.\end{aligned}$$

Furthermore

$$\overline{u}_r = \frac{1}{2\pi}\int_{C_r} u_r\,d\theta,$$

hence

$$|\overline{u}_r|^2 \le \frac{1}{2\pi}\int_{C_r} u_r^2\,d\theta.$$

Therefore, we obtain

$$\begin{aligned}\|r\overline{u}_r\|^2_{L^2(D)} &= 2\pi\int_0^1 |\overline{u}_r|^2\,r^3\,dr\\ &\le \int_0^1\int_{C_r} u_r^2 r\,d\theta\,dr\\ &= \|u_r\|^2_{L^2(D)}.\end{aligned}$$

Thus, we find

$$\|\overline{u}\|^2_{L^2(D)} \leq \frac{1}{2}\|u\|^2_{L^2(\partial D)} + \|\overline{u}\|_{L^2(D)}\|u_r\|_{L^2(D)},$$

which yields

$$\|\overline{u}\|^2_{L^2(D)} \leq \|u\|^2_{L^2(\partial D)} + \|u_r\|^2_{L^2(D)}.$$

Combining this last inequality with (3.6), we conclude

$$\|u\|_{L^2(D)} \leq \|u_r\|_{L^2(D)} + \left\|r^{-1}u_\theta\right\|_{L^2(D)} + \|u\|_{L^2(\partial D)}.$$

This implies (3.3), and completes the proof of the theorem. □

3.2 Dirichlet's Principle

We can now prove the following weak version of Dirichlet's Principle.

Theorem 3.4 *Let $\varphi \in H_1(\Omega)$, then there exists $u \in H_1(\Omega)$ such that $u-\varphi \in H_{1,0}(\Omega)$, and $E(u) \leq E(v)$ for all $v \in H_1(\Omega)$ for which $v - \varphi \in H_{1,0}(\Omega)$.*

Proof. Let $H_\varphi = \{v \in H_1(\Omega);\ v - \varphi \in H_{1,0}(\Omega)\}$. Clearly $H_\varphi \neq \emptyset$ since $\varphi \in H_\varphi$. Let $c = \inf_{H_\varphi} E$. We will show that there exists $u_0 \in H_\varphi$ such that $E(u_0) = c$. Let $u_j \in H_\varphi$ be a sequence such that $E(u_j) \to c$. It follows from Corollary 3.1 that u_j is uniformly bounded in $H_1(\Omega)$, hence we may assume that $u_j \rightharpoonup u_0$ in $H_1(\Omega)$. Since H_φ is closed and convex, it follows from Exercise 3.4 that it is weakly closed, hence $u_0 \in H_\varphi$. The claim now follows, as in the proof of Theorem 3.1, from the weak lower semi-continuity of E on H_φ. □

It remains to analyze the boundary behavior of the solution. Here we restrict ourselves to the case $n = 2, 3$, and prove the following result. Recall that for $n = 2, 3$, $H_2(\Omega) \hookrightarrow C^{1/2}(\Omega)$, hence functions $\varphi \in H_2(\Omega)$ have well defined values $\varphi|\partial D \in C^0(\partial\Omega)$.

Theorem 3.5 *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. Let $\varphi \in H_2(\Omega)$, and define*

$$\mathcal{H}_\varphi = \{v \in C^1(\Omega) \cap C^0(\overline{\Omega});\ v|\partial\Omega = \varphi;\ E(v) < \infty\}.$$

Then there exists a unique $u \in \mathcal{H}_\varphi$ which minimizes E, i.e. $E(u) \leq E(v)$ for every $v \in \mathcal{H}_\varphi$. Furthermore, the solution $u \in C^\infty(\Omega)$, and satisfies in Ω the equation:

$$\Delta u = 0. \tag{3.7}$$

Proof. Note that under our assumptions $\mathcal{H}_\varphi \subset H_\varphi$, where H_φ is as in the proof of Theorem 3.4. By Theorem 3.4, there exists $u \in H_\varphi$ such that u minimizes E over H_φ. Now u satisfies (3.7) weakly, i.e.

$$\int_\Omega \nabla u \cdot \nabla \psi = 0,$$

for every $\psi \in H_{1,0}(\Omega)$. Thus, by Theorem 2.13, $u \in C^\infty(\Omega)$, and satisfies (3.7) in the classical sense. Furthermore, by Theorem 2.14 $u \in H_2(\Omega)$, hence $u \in C^{1/2}(\Omega)$ and in particular $u \in C^0(\overline{\Omega})$. Therefore, by Proposition 2.2, $u|\partial\Omega = \varphi$, and thus $u \in \mathcal{H}_\varphi$. Uniqueness follows from Theorem 2.11, and the fact that every minimizer of E on $\mathcal{H}_\varphi$ satisfies (3.7), as proved in the introduction. □

3.3 A General Variational Principle

Finally, we state for future reference, the following theorem [16]. Its proof, which is essentially the same as that of Theorem 3.4, is left as an exercise.

Theorem 3.6 *Suppose that K is a weakly closed subset of a separable Hilbert space $\mathfrak{H}$. Let*

$$F\colon K \to \mathbb{R}$$

be weakly lower semi-continuous *on K, i.e.*

$$F(x) \leq \liminf F(x_j),$$

whenever $x_j \rightharpoonup x$. Assume that F is coercive*, i.e.*

$$F(x_j) \to \infty,$$

whenever $\|x_j\| \to \infty$. Then, there exists a minimizer *x_0 of F on K, i.e.*

$$F(x_0) \leq F(x),$$

for every $x \in K$.

3.4 Poisson's Equation

In this section, we use the calculus of variation to study Poisson's equation:

$$\Delta u = \rho,$$

with either Dirichlet or Neumann boundary conditions. Then, we apply regularity theory to obtain a classical solution.

Theorem 3.7 *Let $\rho \in C_0^\infty(\Omega)$. Then there exists a unique solution of*

$$\begin{cases} \Delta u &= \rho, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Furthermore, $u \in C^\infty(\overline{\Omega})$.

Remark. Of course, the requirement that ρ be of compact support in Ω is not necessary [6]. We adopt it here only to simplify the analysis at the boundary.

Proof. Consider the functional F defined on $H_{1,0}(\Omega)$ by:

$$F(u) = E(u) + (u,\rho)_{L^2(\Omega)} = \int_\Omega \left\{ \frac{1}{2} |\nabla u|^2 + u\rho \right\}.$$

We have already seen that E is weakly lower semi-continuous on $H_{1,0}(\Omega)$. Since the embedding $H_{1,0}(\Omega) \hookrightarrow L^2(\Omega)$ is compact, it follows that $(u,\rho)_{L^2(\Omega)}$ is weakly continuous on $H_{1,0}(\Omega)$, and hence F is weakly lower semi-continuous on $H_{1,0}(\Omega)$. Also, in view of the Poincaré inequality, Theorem 3.1, F is coercive. Indeed, with c the constant in (3.1), we have

$$\begin{aligned} F(u) &\geq \|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^n)} - c\,\|\rho\|^2_{L^2(\Omega)} - \frac{1}{4c}\,\|u\|^2_{L^2(\Omega)} \\ &\geq \frac{1}{2}\,\|\nabla u\|^2_{L^2(\Omega;\mathbb{R}^n)} - c\,\|\rho\|^2_{L^2(\Omega)} \\ &\geq \frac{1}{2(c+1)}\,\|u\|^2_{H_1(\Omega)} - c\,\|\rho\|^2_{L^2(\Omega)}. \end{aligned}$$

Therefore, we may apply Theorem 3.6, to conclude that there exists a minimizer u of F on $H_{1,0}(\Omega)$, and it satisfies the weak form of (3.8):

$$\int_\Omega \{\nabla u \cdot \nabla v + \rho v\} = 0, \tag{3.9}$$

for any $v \in H_{1,0}(\Omega)$. It remain to show that $u \in C^\infty(\overline{\Omega})$, since we can then integrate by parts and obtain that u satisfies the classical form of (3.8). Let $K = \operatorname{supp}\rho$. Note that u is harmonic in $\Omega\backslash K$, for if $v \in C_0^\infty(\Omega\backslash K)$, then clearly the second term in (3.9) is identically zero. Thus, we need only establish interior regularity. Let $B_1 \subset\subset \Omega$, let $2\varepsilon < \operatorname{dist}(B_1, \partial\Omega)$, and let $u_\varepsilon = S_\varepsilon u$ be defined as before, see (2.5). For $x \in B_1$ we have

$$\begin{aligned}\Delta u_\varepsilon(x) &= \int_\Omega u(y)\,\Delta_x\varphi_\varepsilon(x-y)\,dy\\ &= \int_\Omega u(y)\,\Delta_y\varphi_\varepsilon(x-y)\,dy\\ &= -\int_\Omega \nabla u(y)\cdot\nabla\varphi_\varepsilon(x-y)\,dy\\ &= \int_\Omega \rho(y)\,\varphi_\varepsilon(x-y)\,dy\\ &= \rho_\varepsilon(x),\end{aligned}$$

where $\rho_\varepsilon = S_\varepsilon\rho$. Thus, u_ε satisfies

$$\Delta u_\varepsilon = \rho_\varepsilon \tag{3.10}$$

in B_1. Let $B_2 \subset\subset B_1$. We use Equation (3.10) to estimate u_ε on B_2, uniformly in ε. Recall that $\rho_\varepsilon \to \rho$ in $H_k(\Omega)$ for any $k \geq 0$. Thus for each $k \geq 0$, there is a constant $C(k)$, such that $\|\rho_\varepsilon\|_{H_k(B_1)} \leq C(k)\,\|\rho\|_{H_k(B_1)}$ uniformly in ε. First, we observe, as in the proof of Proposition 2.4, that

$$\|u_\varepsilon\|_{L^2(B_1)} \leq \|u\|_{L^2(\Omega)},$$

and since $\nabla u_\varepsilon = \Big(\nabla u\Big)_\varepsilon$, we also have

$$\|\nabla u_\varepsilon\|_{L^2(B_1;\mathbb{R}^n)} \leq \|\nabla u\|_{L^2(\Omega;\mathbb{R}^n)}. \tag{3.11}$$

We will derive the H_2 estimate; the rest is similar. Let $\psi \in C_0^\infty(B_1)$ be a cut-off function, $\psi = 1$ on B_2, $0 \leq \psi \leq 1$. Differentiate (3.10) in the direction $\vec{e}_j$:

$$\Delta\nabla_{\vec{e}_j} u_\varepsilon = \nabla_{\vec{e}_j}\rho_\varepsilon,$$

multiply by $\psi^2\nabla_{\vec{e}_j}u_\varepsilon$, sum over $j = 1, \ldots, n$, and integrate over B_1, to get

$$\int_{B_1}\psi^2\left|\nabla^2 u_\varepsilon\right|^2 = -2\int_{B_1}\psi\,\nabla^2 u_\varepsilon\cdot\nabla u_\varepsilon\cdot\nabla\psi + \int_{B_1}\psi^2\,\rho_\varepsilon^2 + 2\int_{B_1}\psi\,\rho_\varepsilon\,\nabla u_\varepsilon\cdot\nabla\psi.$$

Thus, we obtain the inequality:

$$\begin{aligned}\|\psi\nabla^2 u_\varepsilon\|_{L^2(B_1;S^{n\times n})} &\leq 2\,\|\nabla\psi\|_{L^\infty(B_1;\mathbb{R}^n)}\,\|\nabla u_\varepsilon\|_{L^2(B_1;\mathbb{R}^n)}\,\|\psi\nabla^2 u_\varepsilon\|_{L^2(B_1;S^{n\times n})}\\ &\quad + \|\psi\rho_\varepsilon\|_{L^2(B_1)} + 2\,\|\nabla\psi\|_{L^\infty(B_1;\mathbb{R}^n)}\,\|\rho_\varepsilon\|_{L^2(B_1)}\,\|\psi\nabla u_\varepsilon\|_{L^2(B_1;\mathbb{R}^n)}.\end{aligned}$$

Together with (3.11), this implies a uniform bound:

$$\left\|\nabla^2 u_\varepsilon\right\|_{L^2(B_2)} \leq C(d)\Big(\|u\|_{H_1(\Omega)} + \|\rho\|_{L^2(\Omega)}\Big), \tag{3.12}$$

where the constant $C(d)$ depends only on $d = \operatorname{dist}(B_2, \partial B_1)$. It follows that u_ε is uniformly bounded in $H_2(B_2)$, hence a subsequence converges weakly in $H_2(B_2)$, and since this subsequence converges strongly in $H_1(\Omega)$ to u, it follows that $u_\varepsilon \rightharpoonup u$ in $H_2(B_2)$, hence $u \in H_2(B_2)$. In fact, with a little extra work, one can show directly that $u_\varepsilon \to u$ strongly in $H_2(B_2)$. Furthermore, by the weak lower semi-continuity of the norm $\|\cdot\|_{H_2(B_2)}$, the Inequality (3.12) holds also for u. Similarly, bounds for u in $H_k(\Omega')$ can be obtained for any $\Omega' \subset\subset \Omega$, and any $k \geq 0$. It follows from the Sobolev inequalities that $u \in C^\infty(\Omega)$. As mentioned earlier, boundary regularity follows from Theorem 2.14. Uniqueness follows, as in the case $\rho = 0$, from Theorem 2.11. $\square$

A similar theorem can be stated and proved for Neumann boundary conditions.

Theorem 3.8 *Let $\rho \in C_0^\infty(\Omega)$ satisfy $\int_\Omega \rho = 0$. Then there exists a solution of*

$$\begin{cases} \Delta u &= \rho, \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{3.13}$$

The solution is unique up to a constant, and thus can be normalized by the condition $\int_\Omega u = 0$. Furthermore, the solution $u \in C^\infty(\overline{\Omega})$.

The proof is left as an exercise.

3.5 Exercises

Exercise 3.1 Prove Corollary 3.1.

Exercise 3.2 Prove Theorem 3.2.

Exercise 3.3 Let $S^1 \subset \mathbb{R}^2$ be the unit circle, and let $u \in H_1(S^1)$ satisfy

$$\int_{S^1} u\, d\theta = 0.$$

Show that

$$\int_{S^1} u^2\, d\theta \leq \int_{S^1} u_\theta^2\, d\theta.$$

Exercise 3.4 Let K be a closed and convex subset of a Hilbert space $\mathfrak{H}$, then K is also weakly closed. For the Banach space version of this theorem, see [14].

Exercise 3.5 Prove Theorem 3.6.

Exercise 3.6 Prove Theorem 3.8.

Chapter 4

Plateau's Problem

In this chapter we prove the existence of a solution to Plateau's problem. Sections 4.1 and 4.3 follow the presentation in [16]. Throughout, $\Gamma \subset \mathbb{R}^n$ is a simple closed Jordan curve. Introduce the space of *parameterized surfaces* spanning Γ:

$$\mathcal{H}_\Gamma = \{X \in H_1(D;\mathbb{R}^n);\ X|\partial D \in C^0(D;\mathbb{R}^n) \text{ is a monotone parameterization of } \Gamma\}$$

with the norm induced from $H_1(D;\mathbb{R}^n)$. Here $X\colon \partial D \to \Gamma$ is *monotone* if, for every $P \in \Gamma$, $X^{-1}(P)$ is connected in ∂D. An equivalent characterization is that the inverse image of every compact connected set in Γ is connected, see Exercise 4.5. X is *strictly monotone* if $X^{-1}(P)$ is a single point for every $P \in \Gamma$. The functionals A and E, as defined by (1.1) and (1.7) respectively, are well-defined and continuous on $\mathcal{H}_\Gamma$, see Exercise 4.1. We will first show that if $\mathcal{H}_\Gamma \neq \emptyset$, then there exists a conformally parameterized minimal surface spanning Γ. This is achieved by minimizing E over $\mathcal{H}_\Gamma$. We also give conditions on Γ for $\mathcal{H}_\Gamma$ to be non-empty. Finally, we prove an 'almost uniformization' result for C^2 Jordan curves, which allows us to conclude that the solution is a surface of least area, i.e. it minimizes A as well.

4.1 Existence of a Minimal Surface Spanning Γ

Definition 4.1 *A map $X \in C^2(D;\mathbb{R}^n)$ is a* conformally parameterized minimal surface *(CPMS), if X is* harmonic, *i.e.*

$$\Delta X = 0, \tag{4.1}$$

and X is conformal, *i.e.*

$$|X_u|^2 = |X_v|^2, \qquad X_u \cdot X_v = 0. \tag{4.2}$$

For an arbitrary domain $G \subset \mathbb{R}^2$, and $X \in H_1(G;\mathbb{R}^n)$, denote

$$E(X;G) = \int_G |\nabla X|^2 .$$

We note that E is invariant under conformal maps.

Lemma 4.1 *Let $X \in H_1(G;\mathbb{R}^n)$, and let $g\colon G \to G'$ be a conformal map. Then*

$$E(X\circ g^{-1};G') = E(X;G).$$

For the proof see Exercise 1.3.

Definition 4.2 *A map $X \in \mathcal{H}_\Gamma$ is* critical *if it satisfies*

$$\frac{d}{dt}E(X+tY)\bigg|_{t=0} = 0, \qquad \forall Y \in H_{1,0}(D;\mathbb{R}^n), \tag{4.3}$$

and

$$\frac{d}{dt}E\Big(X\circ g_t^{-1};D(t)\Big)\bigg|_{t=0} = 0, \tag{4.4}$$

for any differentiable 1-parameter family of diffeomorphisms $g_t\colon D \to D(t)$, with $g_0 = \mathrm{id}$.

We first make the following simple observation.

Lemma 4.2 *If X minimizes E over $\mathcal{H}_\Gamma$ then X is critical.*

Proof. Indeed, X satisfies (4.3) since X in particular minimizes E over the subspace $X + H_{1,0}(D) \subset H_1(D;\mathbb{R}^n)$. Suppose that X does not satisfy (4.4) for some family g_t. Then for some diffeomorphism $g\colon D \to D' \subset \mathbb{R}^2$,

$$E(X\circ g^{-1};D') < E(X),$$

and $g|\partial D$ is monotone. By the Riemann mapping theorem, there is a conformal diffeomorphism $f\colon D \to D'$, and $f|\partial D$ is monotone. It follows that $X\circ g^{-1}\circ f \in \mathcal{H}_\Gamma$. By the conformal invariance of E, we conclude that

$$E(X\circ g^{-1}\circ f) = E(X\circ g^{-1};D') < E(X),$$

which contradicts the minimizing property of X. □

The two notions given by Definitions 4.1 and 4.2 are equivalent as stated in the following theorem.

Theorem 4.1 *$X \in \mathcal{H}_\Gamma$ is a CPMS if and only if X is critical.*

Remark. Thus, by Lemma 4.2, we will have found a CPMS spanning Γ if we can minimize E over $\mathcal{H}_\Gamma$.

Proof. Since

$$\frac{d}{dt}E(X+tY)\bigg|_{t=0} = \int_D \nabla X \cdot \nabla Y,$$

X satisfies (4.3) if and only if X is weakly harmonic. Hence, by Theorem 2.13, X satisfies (4.3) if and only if X is harmonic. It remains to show that X satisfies (4.4) if and only if X is conformal. Suppose first that X satisfies (4.4). Let $\tau \in C^1(\overline{D};\mathbb{R}^2)$, then for $|t|$ small enough, $g_t = \mathrm{id} + t\tau$ is a diffeomorphism, see Exercise 4.2. Write $\tau = (\tau^1, \tau^2)$, then one calculates:

$$\frac{d}{dt}E\Big(X\circ g_t^{-1}; D(t)\Big)\bigg|_{t=0} = -\frac{1}{2}\int_D \Big\{(e-f)\Big(\tau_u^1 - \tau_v^2\Big) + 2g\Big(\tau_v^1 + \tau_u^2\Big)\Big\}, \tag{4.5}$$

where, as usual, $e = |X_u|^2$, $f = |X_v|^2$, $g = X_u \cdot X_v$, see Exercise 4.3. We introduce complex notation to simplify the formulae. Let $\zeta = u + iv$, $\tau = \tau^1 + i\tau^2$,

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\,\frac{\partial}{\partial v}\right), \qquad \overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right),$$

and

$$\Phi = 4\Big(\partial X\Big)^2 = (e-f) - 2ig.$$

Then, we find

$$(e-f)\Big(\tau_u^1 - \tau_v^2\Big) + 2g\Big(\tau_v^1 + \tau_u^2\Big) = \Re\Big(\Phi\,\overline{\partial}\tau\Big).$$

Thus, Equation (4.5) now reads:

$$\int_D \Re\Big(\Phi\,\overline{\partial}\tau\Big) = 0, \qquad \forall \tau \in C^1(\overline{D};\mathbb{C}). \tag{4.6}$$

We claim that this implies $\Phi = 0$, and hence X is conformal. In order to achieve this, we will need the following lemma.

Lemma 4.3 *Let $h \in C_0^\infty(D;\mathbb{C})$. Then there exists $\tau \in C^\infty(\overline{D};\mathbb{C})$ satisfying*

$$\overline{\partial}\tau = h \tag{4.7}$$

Proof. Note that $\partial\overline{\partial} = \overline{\partial}\partial = (1/4)\Delta$. Now, by Theorem 3.7, there exists a unique $\sigma \in C^\infty(\overline{D};\mathbb{C})$ satisfying

$$\begin{cases} \Delta\sigma &= 4h, \quad \text{in } D, \\ \sigma &= 0, \quad \text{on } \partial D. \end{cases}$$

Set $\tau = \partial\sigma$, then $\tau \in C^\infty(\overline{D};\mathbb{C})$ and $\overline{\partial}\tau = \overline{\partial}\partial\sigma = h$. This completes the proof of the lemma. □

We now return to the proof of the theorem. First note that Φ is harmonic. Indeed, for any $\varphi \in C_0^\infty(D;\mathbb{C})$, we can, by Lemma 4.3, solve

$$\overline{\partial}\tau = \Delta\varphi.$$

Thus, substituting this τ in (4.6), we obtain

$$\int_D \Re\Big(\Phi\,\Delta\varphi\Big) = 0, \qquad \forall\varphi \in C_0^\infty(D;\mathbb{C}).$$

This show that Φ is weakly harmonic. By a variant of Theorem 2.13, see [8], it follows that $\Phi \in C^\infty(D;\mathbb{C})$. We remark here, that we could have obtained this last result from the fact that X is harmonic. We avoided this approach in order to prove that in fact Equation (4.4) holds if and only if X is conformal. Now let $\chi \in C_0^\infty(D)$ be a non-negative cut-off function, then by Lemma 4.3, we can solve

$$\overline{\partial}\tau = \chi\overline{\Phi},$$

Substituting this τ in (4.6), we obtain

$$\int_D \chi\,|\Phi|^2 = 0.$$

Thus $\Phi = 0$ on $\operatorname{supp}\chi$, and this implies that $\Phi = 0$ on D. Now suppose that X is conformal. Let $g_t\colon D \to D(t)$ be a differentiable 1-parameter family of diffeomorphisms, then replacing τ with the vector field $dg_t/dt|_{t=0}$, one sees that Equation (4.5) still holds. Thus, if X is conformal, then X satisfies (4.4). This completes the proof of the theorem. □

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We make the following simple remark whose proof is immediate.

Lemma 4.4 *Let $X \in C^2(D;\mathbb{R}^n)$ be harmonic. Then $\Phi = 4\Big(\partial X\Big)^2$ is holomorphic, i.e.*

$$\overline{\partial}\Phi = 0.$$

We now turn our attention to the conformal group of the disk. Let

$$\mathfrak{G} = \left\{ g(\zeta) = e^{i\phi}\frac{\omega + \zeta}{1 + \overline{\omega}\zeta};\ \omega \in D,\ \phi \in \mathbb{R} \right\}.$$

denote the Mobiüs group, the group of conformal transformations of the unit disk. The invariance of E under the non-compact group $\mathfrak{G}$ causes difficulties, as the following result shows.

Proposition 4.1 *Let $X \in \mathcal{H}_\Gamma$ and denote by*

$$X \circ \mathfrak{G} = \{X \circ g;\ g \in \mathfrak{G}\}$$

the orbit of X under $\mathfrak{G}$. Then the weak closure of $X \circ \mathfrak{G}$ in $H_1(D; \mathbb{R}^n)$ contains a constant map.

Proof. Consider first $\varphi \in C^1(\overline{D}; \mathbb{R}^n)$. Let

$$g_j(\zeta) = \frac{\omega_j + \zeta}{1 + \overline{\omega}_j \zeta},$$

with $\omega_j \in D$, and $\omega_j \to 1$. Clearly, $g_j(\zeta) \to 1$ for all $\zeta \in D$. Thus, $\varphi_j = \varphi \circ g_j \to \varphi(1)$, pointwise in D. Now $E(\varphi_j) = E(\varphi)$, and $\|\varphi_j\|_{L^\infty(D)} = \|\varphi\|_{L^\infty(D)}$, hence φ_j is uniformly bounded in $H_1(D)$, and thus we may assume that $\varphi_j \rightharpoonup \varphi_0 = \varphi(1)$. This proves the proposition for $X \in C^1(\overline{D}; \mathbb{R}^n)$. Now take $X \in \mathcal{H}_\Gamma$, and let $X_j = X \circ g_j^{-1}$. Again $E(X_j) = E(X)$, and by the Poincaré inequality, Theorem 3.3, we have

$$\|X_j\|_{L^2(D)} \le C\left\{E(X_j)^{1/2} + \|X_j\|_{L^2(\partial D)}\right\} \le C'\left\{E(X)^{1/2} + \|X_j\|_{L^\infty(\partial D)}\right\}. \tag{4.8}$$

Thus X_j is uniformly bounded in $H_1(D; \mathbb{R}^n)$, and we may assume that $X_j \rightharpoonup X_0$ in $H_1(D; \mathbb{R}^n)$. It remains to show that X_0 is constant, and to do this it is enough to show that

$$\int_D \nabla X_0 \cdot \nabla \varphi = 0, \qquad \forall \varphi \in C^1(\overline{D}; \mathbb{R}^n). \tag{4.9}$$

Indeed, by approximation, Equation (4.9) will then hold also for all $\varphi \in H_1(D; \mathbb{R}^n)$, and setting $\varphi = X_0$ will yield the result. Note that the quadratic form

$$E(X, Y) = \int_D \nabla X \cdot \nabla Y$$

is also invariant under $\mathfrak{G}$, since

$$E(X,Y)=\frac{1}{2}\Big(E(X+Y)-E(X)-E(Y)\Big).$$

Let $\varphi\in C^1(\overline{D};\mathbb{R}^n)$, and set $\varphi_j=\varphi\circ g_j$. Then by the first part of the proof $\varphi_j\rightharpoonup\varphi_0=$ const. in $H_1(D;\mathbb{R}^n)$, hence

$$\begin{aligned}\int_D \nabla X_0\cdot\nabla\varphi &= \lim_j\int_D \nabla X_j\cdot\nabla\varphi\\ &= \lim_j\int_D \nabla X\cdot\nabla\varphi_j\\ &= \int_D \nabla X\cdot\nabla\varphi_0\\ &= 0.\end{aligned}$$

This completes the proof of the proposition. □

The proof of the following lemma is elementary and is left as an exercise.

Lemma 4.5 *Given two triples (ϕ_1,ϕ_2,ϕ_3), and $(\psi_1,\psi_2\psi_3)$, satisfying $0\le\phi_1<\phi_2<\phi_3<2\pi$, and $0\le\psi_1<\psi_2<\psi_3<2\pi$, there exists a unique $g\in\mathfrak{G}$ such that*

$$g(e^{i\phi_j})=e^{i\psi_j},\qquad j=1,2,3.$$

This allows us to normalize the maps in $\mathcal{H}_\Gamma$ using a *three point condition*. Let ω_j, be three distinct points on ∂D, say

$$\omega_j=e^{i\,2\pi j/3},\qquad j=1,2,3.$$

Let $Q_j\in\Gamma$ be three distinct 'consecutive' points. We define the space of *normalized maps*:

$$\mathcal{H}_\Gamma^*=\{X\in\mathcal{H}_\Gamma;\ X(\omega_j)=Q_j,\ j=1,2,3\}.$$

Lemma 4.6 *For any choice of ω_j and Q_j,*

$$\inf_{\mathcal{H}_\Gamma^*}E=\inf_{\mathcal{H}_\Gamma}E.$$

Proof. Let $X\in\mathcal{H}_\Gamma$. For each $j=1,2,3$, pick $\tau_j\in X^{-1}(Q_j)$. Then, by Lemma 4.5, there exists $g\in\mathfrak{G}$ such that $g(\omega_j)=\tau_j$, and thus $X\circ g\in\mathcal{H}_\Gamma^*$. By the conformal invariance of E, we have:

$$E(X)=E(X\circ g)\ge\inf_{\mathcal{H}_\Gamma^*}E.$$

Taking infimum over all such X yields $\inf_{\mathcal{H}_\Gamma}E\ge\inf_{\mathcal{H}_\Gamma^*}E$. The other inequality is immediate. This completes the proof of the lemma. □

Let $R\colon \mathcal{H}_\Gamma^* \to C^0(\partial D; \mathbb{R}^n)$ be the restriction map, $R\colon X \mapsto X|\partial D$. The key result for the existence problem is the following proposition.

Proposition 4.2 *R is compact.*

Proof. For the proof, we need the Courant-Lebesgue lemma.

Lemma 4.7 *Let $X \in H_1(D; \mathbb{R}^n)$, $\omega_0 \in \overline{D}$, and $\delta \in (0,1)$. Then, there exists $\rho \in [\delta, \sqrt{\delta}]$, such that on $C_\rho(\omega_0) = \partial D_\rho(\omega_0) \cap D$, we have $X_s \in L^2(C_\rho)$, and*

$$\int_{C_\rho} |X_s|^2\, ds \le \frac{4E(X)}{\rho\,|\log \rho|}, \tag{4.10}$$

where s denotes arclength on C_ρ.

Proof. In fact, we prove, by contradiction, that the set

$$\{\rho \in [\delta, \sqrt{\delta}];\ (4.10)\ \text{holds}\}$$

is of positive Lebesgue measure. Assume on the contrary that

$$\int_{C_\rho} |X_s|^2\, ds > \frac{4E(X)}{\rho\,|\log \rho|}, \qquad \text{a.e. } \rho \in [\delta, \sqrt{\delta}]. \tag{4.11}$$

Integrating (4.11) over $[\delta, \sqrt{\delta}]$, we obtain

$$(4\log 2)E(X) < \int_\delta^{\sqrt{\delta}} \int_{C_\rho} |X_s|^2\; ds\, d\rho \le 2E(X),$$

which is clearly a contradiction if $E(X) < \infty$. □

We now return to the proof of the proposition. Let $b > 0$, and define

$$K_b = \{X \in \mathcal{H}_\Gamma^*;\ E(X) \le b\}.$$

We claim that given $\varepsilon > 0$, and $\omega_0 \in \partial D$, then there exists $\delta > 0$, such that for every $X \in K_b$, and every $\omega \in \partial D \cap D_\delta(\omega_0)$, we have

$$|X(\omega) - X(\omega_0)| < 2\varepsilon.$$

It follows that $R(K_b) \subset C^0(\partial D; \mathbb{R}^n)$ is equicontinuous, and hence by the Arzela-Ascoli theorem, $R(K_b)$ is compact. This shows that R is a compact operator. It remains to prove the claim. Choose $\delta_0 > 0$, small enough so that any disk $D_{\sqrt{\delta_0}}$ of radius $\sqrt{\delta_0}$ contains at most one of the ω_j. Choose ε_0 small enough so that each ball $B_{\varepsilon_0} \subset \mathbb{R}^n$

of radius ε_0 contains at most one of the Q_j. We may assume that $\varepsilon < \varepsilon_0$. Choose $0 < \varepsilon_1 < \varepsilon$, such that for any two points $P, S \in \Gamma$ with $|P - S| < \varepsilon_1$, there is a subarc $\Gamma' \subset \Gamma$ joining P and S, contained in a ball of radius ε. To prove this is always possible, assume on the contrary, that there exists sequences P_j, S_j with $|P_j - S_j| \to 0$, and any subarc joining P_j and S_j intersects $\partial B_\varepsilon(P_j)$. Since Γ is compact, we may assume that $P_j, S_j \to P \in \Gamma$. Pick Z a continuous strictly monotone parameterization of Γ, and let $P_j = Z(\xi_j)$, $S_j = Z(\eta_j)$. Then, we have that $\xi_j, \eta_j \to \xi \in \partial D$, and by continuity $Z(\xi) = P$. Let C_j be the shorter subarc of ∂D joining ξ_j and η_j. By assumption, there exists $\tau_j \in C_j$ such that $Z(\tau_j) = T_j \in \Gamma \cap \partial B_\varepsilon(P_j)$. Now, $\tau_j \to \xi$, and possibly going to a subsequence, $T_j \to T \in \Gamma$ with $|T - P| = \varepsilon$. By continuity $Z(\xi) = T$, in contradiction to $Z(\xi) = P$. This justifies the possibility to choose ε_1. Since $\varepsilon_1 < \varepsilon_0$, for each $P, S \in \Gamma$, with $|P - S| < \varepsilon_1$, there exists a unique subarc $\Gamma' \subset \Gamma$ joining P and S characterized by the condition that Γ' contains at most one of the Q_j. Now choose $0 < \delta \leq \delta_0$ such that

$$8\pi b < \varepsilon_1^2 \left|\log \delta\right|.$$

Let $X \in K_b$, $\omega_0 \in \partial D$, and choose using Lemma 4.7 $\rho \in [\delta, \sqrt{\delta}]$ such that

$$\int_{C_\rho} |X_s|^2 \, ds \leq \frac{4E(X)}{\rho \left|\log \rho\right|}.$$

Let $\xi_j \in \partial D \cap C_\rho$, $j = 1, 2$, be the intersection points of ∂D with C_ρ, and C'_ρ the subarc of ∂D which contains at most one of the ω_j. Let $X_j = X(\xi_j)$, and $\Gamma' = X(C'_\rho)$. Then Γ' is the unique subarc of Γ which contains at most one of the Q_j. Now, we have

$$\begin{aligned}
|X_1 - X_2|^2 &\leq \left(\int_{C_\rho} |X_s| \, ds\right)^2 \\
&\leq \pi\rho \int_{C_\rho} |X_s|^2 \, ds \\
&\leq \frac{4\pi E(X)}{|\log \rho|} \\
&\leq \frac{8\pi E(X)}{|\log \delta|} \\
&< \varepsilon_1^2.
\end{aligned}$$

It follows that Γ' is contained in a ball B_ε of radius ε. In particular, for any $\omega \in C'_\delta = \partial D \cap D_\delta(\omega_0)$, we have $\omega \in C'_\rho$, and $X(\omega), X(\omega_0) \in \Gamma' \subset B_\varepsilon$. Therefore

$$|X(\omega) - X(\omega_0)| < 2\varepsilon,$$

and the claim is proved. This completes the proof of the proposition. $\square$

Lemma 4.8 *Let $X_j \in C^0(\partial D; \mathbb{R}^n)$ be a sequence of continuous monotone parameterizations of a simple closed Jordan curve Γ, and suppose that $X_j \to X$ uniformly on ∂D. Then $X \in C^0(\partial D; \mathbb{R}^n)$ is also a continuous monotone parameterization of Γ.*

Proof. Let $Z \colon \partial D \to \mathbb{R}^n$ be a fixed strictly monotone parameterization of Γ. Then $Z^{-1} \colon \Gamma \to \partial D$ is continuous and strictly monotone. Let

$$\sigma_j = Z^{-1} \circ X_j \colon \ \partial D \to \partial D.$$

Then, σ_j is continuous monotone, and by the continuity of Z^{-1}, converges pointwise on ∂D to $\sigma = Z^{-1} \circ X$. Since X is continuous, σ is continuous, and it follows from Exercise 4.6, that σ is monotone. Therefore, we conclude that $X = Z \circ \sigma$ is monotone. This completes the proof of the lemma. □

Everything now follows from Proposition 4.2.

Proposition 4.3 *The set $K_b = \{X \in \mathcal{H}^*_\Gamma;\ E(X) \leq b\}$ is weakly closed in $H_1(D; \mathbb{R}^n)$.*

Proof. Let $X_j \in K_b$ converge weakly to $X \in H_1(D, \mathbb{R}^n)$. Then, by the weak lower semi-continuity of E, $E(X) \leq b$, and by Proposition 4.2, there is a subsequence $X_j \to X$ uniformly on ∂D. By Lemma 4.8, X is a continuous monotone parameterization of Γ, hence $X \in \mathcal{H}^*_\Gamma$. □

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The main result of this section follows.

Theorem 4.2 *Let $\Gamma \subset \mathbb{R}^n$ be a simple closed Jordan curve such that $\mathcal{H}_\Gamma \neq \emptyset$. Then there is a CPMS spanning Γ.*

Proof. Since $\mathcal{H}_\Gamma \neq \emptyset$, it follows as in the proof of Lemma 4.6 that $\mathcal{H}^*_\Gamma \neq \emptyset$. The set $\mathcal{H}^*_\Gamma$ is weakly closed by Proposition 4.3. The functional E is weakly lower semi-continuous, and by Poincaré inequality, Theorem 3.3, E is also coercive; see the proof of Proposition 4.1. Thus, by Theorem 3.6 there exists a minimizer X_0 of E on $\mathcal{H}^*_\Gamma$. By Lemma 4.6, X_0 minimizes E on $\mathcal{H}_\Gamma$. By Lemma 4.2, X_0 is critical. Finally, by Theorem 4.1, X_0 is a CPMS. □

We conclude this section with the following result on the boundary behavior of any solution to Plateau's problem.

Theorem 4.3 *Let Γ be a simple closed Jordan curve, and let $X \in \mathcal{H}_\Gamma$ be a CPMS. Then $X|\partial D \to \Gamma$ is a homeomorphism, i.e. $X|\partial D$ is strictly monotone.*

Proof. Assume on the contrary that $P = X(\omega) = X(\tau)$ for $\omega \neq \tau$ on ∂D. We may assume that $P = 0$, the origin in $\mathbb{R}^n$. By the monotonicity of $X|\partial D$, it follows that $X(\xi) = 0$ for all $\xi \in C$ an arc of ∂D joining ω and τ. Let $\mathbb{C}_+ = \{\zeta \in \mathbb{C};\ \Im\zeta > 0\}$ be the upper half plane in $\mathbb{C}$, and $\psi\colon \mathbb{C}_+ \to D$ a conformal diffeomorphism with $\psi^{-1}(C) = I = [0,1] \subset \mathbb{R}$. Then, $Y = X\circ\psi\colon \mathbb{C}_+ \to \mathbb{R}^n$ is harmonic, and $Y|I = 0$. Define an extension of Y to $G = \{\zeta \in \mathbb{C};\ 0 < \Re\zeta < 1\}$, by

$$Y(\overline{\zeta}) = -Y(\zeta),$$

for $\zeta \in G\backslash\mathbb{C}_+$. Then Y is smooth in G, harmonic on $G\backslash I$, hence harmonic everywhere in G. Since the zero of a non-constant harmonic map are isolated, it follows from $Y|I = 0$, that $Y \equiv 0$ in G, hence $Y \equiv 0$ in $\mathbb{C}_+$, hence $X \equiv 0$ in D. This is in contradiction to $X(\partial D) = \Gamma$. $\square$

4.2 The Isoperimetric Inequality

In this section we prove the isoperimetric inequality. One of its consequences is a general condition for $\mathcal{H}_\Gamma$ to be non-empty.

Proposition 4.4 *Let A be the area of a conformally parameterized minimal surface $X \in C^1(\overline{D}, \mathbb{R}^n)$, and let L be the length of its perimeter. Then, the following inequality holds*

$$A \leq \frac{L^2}{4\pi}, \tag{4.12}$$

with equality if and only if X represents a disk. In other words, among all minimal surfaces of the same perimeter, the disk has the largest area.

Proof. Let (r,θ) be polar coordinates on D. We may assume that the origin of $\mathbb{R}^n$ is chosen at the center of gravity of Γ, i.e.

$$\int_0^{2\pi} X\, ds = 0$$

where s is arclength along Γ. Since X is conformal and harmonic, we have, integrating (1.7) by parts:

$$A(X) = E(X) = \frac{1}{2}\int_D |\nabla X|^2 = \frac{1}{2}\int_{\partial D} X \cdot X_r\, d\theta.$$

Now, the conformal relations, Equations (4.2), imply that

$$|X_r|^2 = \left|\frac{1}{r}X_\theta\right|^2.$$

Thus, since $r = 1$ on ∂D, we find

$$A \leq \frac{1}{2} \int_{\partial D} |X| \; |X_\theta| \, d\theta.$$

Note that this integral is re-parameterization invariant. Let $t = 2\pi s/L$. Then, we obtain

$$A \leq \frac{1}{2} \int_0^{2\pi} |X| \; |X_t| \Big|_{r=1} dt.$$

Furthermore, on ∂D, $|X_t| = L/2\pi$, hence

$$\frac{L^2}{2\pi} = \int_0^{2\pi} |X_t|^2 \Big|_{r=1} dt.$$

Thus, we find

$$\begin{aligned} \frac{L^2}{2\pi} - 2A &\geq \int_0^{2\pi} \left\{ |X_t|^2 - |X| \; |X_t|^2 \right\} \Big|_{r=1} dt \\ &= \frac{1}{2} \int_0^{2\pi} \left\{ (|X_t| - |X|)^2 + |X_t|^2 - |X|^2 \right\} \Big|_{r=1} dt. \end{aligned}$$

It remains to show that

$$\int_0^{2\pi} \left\{ |X_t|^2 - |X|^2 \right\} \Big|_{r=1} dt \geq 0, \tag{4.13}$$

for all $X \in C^1(\partial D, \mathbb{R}^n)$. This follows from Poincaré inequality, Exercise 3.3, since $\int_0^{2\pi} X \, dt = 0$. If equality holds in (4.13), then

$$X = a \cos t + b \sin t,$$

in which case Γ is a circle and X represents a disk. This completes the proof of the proposition. □

We now give a criterion for $\mathcal{H}_\Gamma$ to be non-empty. Let S^1 denote the unit circle in $\mathbb{R}^2$.

Definition 4.3 *A curve* Γ *is* rectifiable, *if it admits a parameterization by a map* $Z \in W^{1,1}(S^1; \mathbb{R}^n)$.

Remark. We have not defined the space $W^{1,1}(S^1)$. It is a good open-ended exercise to define, and study some of the properties of $W^{1,p}(S^1)$. In particular, the reader should try to show that $C^\infty(S^1)$ is dense in $W^{1,p}(S^1)$.

Proposition 4.5 *Let Γ be a rectifiable simple closed Jordan curve. Then $\mathcal{H}_\Gamma \neq \emptyset$.*

Proof. Let Z be a continuous strictly monotone parameterization of Γ. Approximate Z by a sequence of smooth maps $Z_k \in C^\infty(S^1;\mathbb{R}^n)$, such that $Z_k \to Z$ in $W^{1,1}(S^1;\mathbb{R}^n)$. Let $Z_k' \in C^\infty(S^1;\mathbb{R}^{n+2})$ be defined by

$$Z_k'(\omega) = \left(Z_k(\omega), \frac{1}{k}\omega\right).$$

Then, also $Z_k' \to Z$ in $W^{1,1}(S^1;\mathbb{R}^{n+2})$, and in particular $Z_k' \to Z$ uniformly on S^1. Here $\mathbb{R}^n$ is considered to be a linear subspace of $\mathbb{R}^{n+2}$. Let $\Gamma_k = Z_k'(S^1)$, and L_k the length of Γ_k, then there is a uniform bound $L_k \leq C$, where C depends only on Γ. By Theorem 3.5 there is a harmonic surface $X_k \in H_1(D;\mathbb{R}^{n+2})$, with boundary conditions $X_k = Z_k$ on ∂D. Hence $\mathcal{H}_{\Gamma_k} \neq \emptyset$, and by Theorem 4.2, there exists CPMS $X_k' \in \mathcal{H}_{\Gamma_k}$. Furthermore, by Theorem 4.4, we have for each k:

$$E(X_k') = A(X_k') \leq \frac{L_k^2}{4\pi} \leq C,$$

where C is independent of k. Let $Q_{k,j} = Z_k(\omega_j)$, $j = 1,2,3$, and re-normalize X_k' by the three points condition with points $Q_{k,j}$, see Lemma 4.6. We now repeat the proof of Proposition 4.2 to show that the sequence $X_k' \in C^0(\partial D;\mathbb{R}^{n+2})$ is pre-compact. The choice of ε_0 uniformly in k is possible since $Q_{k,j} \to Q_j = Z(\omega_j)$. It remains to check that $\varepsilon_1 > 0$ can be chosen uniformly in k. Suppose this is impossible. Then there are points P_k, $S_k \in \Gamma_k$, with $P_k, S_k \to P \in \Gamma$, and any arc of Γ_k joining P_k and S_k intersects $\partial B_\varepsilon(P_k)$. Let $P_k = Z_k(\xi_k)$, and $S_k = Z_k(\eta_k)$, then $\xi_k, \eta_k \to \xi \in \Gamma$, and by assumption, there is $\tau_k \in C_k$, the shorter arc of ∂D joining ξ_k and η_k, such that $T_k = Z_k(\tau_k) \in \partial B_\varepsilon(P_k)$. We may assume that $T_k \to T \in \Gamma$, and clearly $\tau_k \to \xi$. Thus we obtain, using the uniform convergence of Z_k, $Z(\xi) = P$, and also $Z(\xi) = T$, which is a contradiction since $|P - T| = \varepsilon$. The rest of the proof is the same. We conclude that a subsequence X_k' converges uniformly on the boundary. It then follows from the maximum principle, Theorem 2.10, that X_k' converges uniformly in $\overline{D}$, and therefore from Harnack's inequality, Proposition 2.3, that all derivatives of X_k' converge uniformly on compact subsets of D. Thus, the limit X is harmonic, $X(\partial D) \subset \Gamma$, and the maximum principle implies that $X(\overline{D}) \subset \mathbb{R}^n$. It remains to check that X is monotone. This will follow from a variant of Lemma 4.8.

Lemma 4.9 *Let Z_k be a sequence of continuous strictly monotone parameterizations of simple closed Jordan curves Γ_k, which converge uniformly on ∂D to a continuous strictly monotone parameterization of a simple closed Jordan curve Γ. Let X_k be continuous parameterizations of Γ_k which converge uniformly on ∂D to X. Then X is a continuous monotone parameterization of Γ.*

Proof of lemma. $Z_k^{-1}\colon \Gamma_k \to \partial D$ is continuous and strictly monotone. Thus,

$$\sigma_k = Z_k^{-1}\circ X_k\colon\ \partial D \to \partial D,$$

is continuous and monotone. Furthermore, σ_k converges pointwise to $\sigma = Z^{-1}\circ X$, and σ is continuous. Indeed, suppose that there is $\omega \in \partial D$, and a subsequence $\sigma_{k_j}(\omega)$ which does not converge to $Z^{-1}\circ X(\omega)$. By compactness, we may assume that $\sigma_{k_j}(\omega) \to \tau \neq Z^{-1}\circ X(\omega)$. But then, by the uniform convergence of Z_{k_j}, $X_{k_j}(\omega) = Z_{k_j}\circ\sigma_{k_j}(\omega) \to Z(\tau)$, and since Z is strictly monotone, $Z(\tau) \neq X(\omega)$. This contradiction shows that $\sigma_k \to \sigma$. By Exercise 4.6, σ is monotone, and hence $X = Z\circ\sigma$ is also monotone. This completes the proof of the lemma. □

We note that without the assumption on the Z_k converging uniformly to Z, the lemma is false. We see that X is monotone, and hence $X \in \mathcal{H}_\Gamma$. Thus $\mathcal{H}_\Gamma \neq \emptyset$, and the proposition is proved. □

In view of this result, we can now restate Theorem 4.2.

Theorem 4.4 *Let Γ be a rectifiable simple closed Jordan curve in $\mathbb{R}^n$. Then there exists a conformally parameterized minimal surface spanning Γ.*

4.3 Uniformization

In this section we prove that minimizing E is equivalent to minimizing A. For simplicity, we restrict ourselves to the case where Γ is of class C^2. Define the class of normalized diffeomorphisms:

$$\mathcal{F} = \{\psi \in C^2(\overline{D};\mathbb{R}^2);\ \psi \text{ is an orientation preserving diffeomorphism of } \overline{D};\\ \psi(\omega_j) = \omega_j,\ j = 1,2,3\}.$$

Let $\overline{\mathcal{F}}$ be the weak closure of $\mathcal{F}$ in $H_1(D;\mathbb{R}^2)$.

Lemma 4.10 *Let $\psi \in \overline{\mathcal{F}}$, then ψ is the uniform limit of a sequence $\psi_k \in \mathcal{F}$. In particular, $\overline{\mathcal{F}} \subset C^0(\overline{D};\mathbb{R}^2) \cap \mathcal{H}^*_{\partial D}$.*

Proof. We claim that bounded subsets of $\mathcal{F}$ are equicontinuous. The lemma follows from this claim by the Arzela-Ascoli theorem. To prove the claim, let

$$\mathcal{F}_b = \{\psi \in \mathcal{F};\ E(\psi) \le b\}.$$

We will show that $\mathcal{F}_b$ is equicontinuous. Let $\zeta_0 \in \overline{D}$, and $\varepsilon > 0$. We may assume that ε is small enough to assure that any disk $D_\varepsilon(\omega)$ of radius ε contains at most one ω_j. Pick $0 < \delta < \varepsilon^2$ so that

$$16\pi b < \varepsilon^2 \left|\log\delta\right|.$$

Let $\psi \in \mathcal{F}_b$. Then by the Courant-Lebesgue Lemma, Lemma 4.7, there is $\rho \in [\delta, \sqrt{\delta}]$, such that

$$\begin{aligned}
\sup_{\zeta,\zeta' \in C_\rho(\zeta_0)} |\psi(\zeta) - \psi(\zeta')|^2 &\leq 2\pi\rho \int_{C_\rho(\zeta_0)} |\psi_s|^2\, ds \\
&\leq \frac{8\pi E(\psi)}{|\log \rho|} \\
&\leq \frac{16\pi b}{|\log \delta|} \\
&< \varepsilon^2,
\end{aligned}$$

where $C_\rho(\zeta_0) = \partial D_\rho(\zeta_0) \cap \overline{D}$. Thus the diameter of $\psi\big(C_\rho(\zeta_0)\big)$ is less than ε, i.e. $\psi\big(C_\rho(\zeta_0)\big)$ is contained in a disk $D_\varepsilon(\omega_0)$ of radius ε, which consequently contains at most one ω_j. On the other hand, $D_\rho(\zeta_0) \cap \overline{D}$ is contained in a disk of radius $\sqrt{\delta} < \varepsilon$, and hence contains at most one ω_j. Therefore, since ψ is a normalized diffeomorphism of $\overline{D}$, $\psi\big(D_\rho(\zeta_0) \cap \overline{D}\big) \subset D_\varepsilon(\omega_0)$. We conclude that if $\zeta \in D_\delta(\zeta_0) \cap \overline{D}$, then $\zeta \in D_\rho(\zeta_0) \cap \overline{D}$, we have $\psi(\zeta), \psi(\zeta_0) \in D_\varepsilon(\omega_0)$, and hence

$$|\psi(\zeta) - \psi(\zeta_0)| < 2\varepsilon.$$

This proves the claim and the lemma. $\square$

Next, we need a change of variable formula for $\psi \in \overline{\mathcal{F}}$. Let $\vec{e}_1, \vec{e}_2$ be the standard orthonormal basis in $\mathbb{R}^2$. For $f \in H_1(D)$, define the dual of the gradient $*df \in L^2(D; \mathbb{R}^2)$, by:

$$*df = \frac{\partial f}{\partial v}\, \vec{e}_1 - \frac{\partial f}{\partial u} \vec{e}_2.$$

Let $\vec{r} = u\vec{e}_1 + v\vec{e}_2$ be the unit normal on ∂D, and note that

$$*df \cdot \vec{r} = \frac{\partial f}{\partial \theta}. \tag{4.14}$$

Now, for $\psi = (\psi^1, \psi^2) \in \overline{\mathcal{F}}$, let

$$\det(\nabla \psi) = \frac{\partial \psi^1}{\partial u}\, \frac{\partial \psi^2}{\partial v} - \frac{\partial \psi^1}{\partial v}\, \frac{\partial \psi^2}{\partial u},$$

and note that for $\psi \in \mathcal{F}$, we have

$$\det(\nabla \psi) = \operatorname{div}(\psi^1 *d\psi^2).$$

Lemma 4.11 *Let $\varphi \in C^1(\overline{D})$, $\psi \in \overline{\mathcal{F}}$. Then*

$$\int_D \varphi = \int_D \varphi\circ\psi \ \det(\nabla\psi).$$

Proof. Let $\psi_k \in \mathcal{F}$ be a sequence converging weakly to ψ in $H_1(\overline{D};\mathbb{R}^2)$, with $E(\psi_k) \leq C$, and $\psi_k \to \psi$ uniformly on $\overline{D}$. Since φ is continuous on $\overline{D}$, we have

$$\|\varphi\circ\psi - \varphi\circ\psi_k\|_{L^\infty(D)} \to 0,$$

as $k \to \infty$. Thus, since $\det(\nabla\psi_k) \leq 2\,|\nabla\psi_k|^2$, we find

$$\left|\int_D \left\{\varphi\circ\psi \ \det(\nabla\psi_k) - \varphi\circ\psi_k \ \det(\nabla\psi_k)\right\}\right| \leq 2E(\psi_k)\,\|\varphi\circ\psi - \varphi\circ\psi_k\|_{L^\infty(D)} \to 0, \tag{4.15}$$

as $k \to \infty$. Now,

$$\int_D \varphi\circ\psi \ \det(\nabla\psi_k) = -\int_D \nabla(\varphi\circ\psi)\cdot\left(\psi_k^1 * d\psi_k^2\right) + \int_{\partial D} (\varphi\circ\psi)\,\psi_k^1\left(*d\psi_k^2\cdot\vec{r}\right),$$

and

$$\int_D \nabla(\varphi\circ\psi)\cdot\left(\psi_k^1 * d\psi_k^2\right) \to \int_D \nabla(\varphi\circ\psi)\cdot\left(\psi^1 * d\psi^2\right). \tag{4.16}$$

Indeed, $*d\psi_k^2 \rightharpoonup *d\psi^2$ in $L^2(D)$, and therefore

$$\begin{aligned}
\left|\int_D \nabla(\varphi\circ\psi)\cdot\left(\psi^1 * d\psi^2 - \psi_k^1 * d\psi_k^2\right)\right| &\leq \left|\int_D \psi^1\,\nabla(\varphi\circ\psi)\cdot\left(*d\psi^2 - *d\psi_k^2\right)\right| \\
&\quad + \left|\int_D (\psi^1 - \psi_k^1)\,\nabla(\varphi\circ\psi)\cdot *d\psi_k^2\right| \\
&\leq \left|\int_D \psi^1\,\nabla(\varphi\circ\psi)\cdot\left(*d\psi^2 - *d\psi_k^2\right)\right| \\
&\quad + \|\nabla\varphi\circ\psi\|_{L^2(D)}\left\|\psi^1 - \psi_k^1\right\|_{L^\infty(D)}\left\|*d\psi_k^2\right\|_{L^2(D)},
\end{aligned}$$

which tends to zero as $k \to \infty$. Finally, as $k \to \infty$,

$$\int_{\partial D} (\varphi\circ\psi)\,\psi_k^1\left(*d\psi_k^2\cdot\vec{r}\right) \to \int_{\partial D} (\varphi\circ\psi)\,\psi^1\left(*d\psi^2\cdot\vec{r}\right), \tag{4.17}$$

where, in view of (4.14), $*d\psi^2\cdot\vec{r}$ is the Lebesgue-Stieltjes measure $d\psi^2$ on ∂D. Note that ψ^2 is of bounded variation, so that the right-hand side of (4.17) makes sense. To prove (4.17), note that $|\partial\psi_k^2/\partial\theta| \le |\partial\psi_k/\partial\theta| \in L^1(\partial D)$, and in fact $\int_{\partial D} |\partial\psi_k/\partial\theta|\, d\theta = 2\pi$. Therefore, we have

$$\left|\int_{\partial D} (\varphi\circ\psi)\left(\psi_k^1 - \psi^1\right) d\psi_k^2\right| \le 2\pi \left\|\varphi\circ\psi\right\|_{L^\infty(\partial D)} \left\|\psi_k^1 - \psi^1\right\|_{L^\infty(\partial D)} \to 0,$$

as $k \to \infty$. Furthermore, since $\varphi \in C^1(\partial D)$, and ψ is continuous and monotone, $(\varphi\circ\psi)\,\psi^1$ is continuous and of bounded variation. Thus, we may integrate by parts

$$\int_{\partial D} (\varphi\circ\psi)\,\psi^1 d\psi_k^2 = -\int_{\partial D} \psi_k^2\, d\Big((\varphi\circ\psi)\,\psi^1\Big).$$

Now, $\psi_k^2 \to \psi^2$ uniformly on ∂D, whence

$$\int_{\partial D} \psi_k^2\, d\Big((\varphi\circ\psi)\,\psi^1\Big) \to \int_{\partial D} \psi^2\, d\Big((\varphi\circ\psi)\,\psi^1\Big) = -\int_{\partial D} (\varphi\circ\psi)\psi^1\, d\psi^2,$$

as $k \to \infty$. Combining (4.15)–(4.17), we conclude that

$$\int_D \varphi = \int_D \varphi\circ\psi_k\, \det(\nabla\psi_k) \to -\int_D \nabla(\varphi\circ\psi)\cdot\Big(\psi^1 * d\psi^2\Big) + \int_{\partial D} (\varphi\circ\psi)\psi^1\Big(* d\psi^2 \cdot \vec{r}\Big),$$

as $k \to \infty$. Integrating the right hand side by parts yields

$$\int_D \varphi = \int_D (\varphi\circ\psi)\, \det(\nabla\psi).$$

This completes the proof of the lemma. □

Lemma 4.12 *Let $Z \in C^2(\overline{D};\mathbb{R}^n)$ with ∇Z non-degenerate, $\psi \in \overline{\mathcal{F}}$. Then*

$$A(Z) = A(Z\circ\psi).$$

Proof. Write $Z = Z(x,y)$, $\psi = \psi(u,v)$, and let

$$\begin{aligned} e &= |Z_x|^2, & f &= |Z_y|^2, & g &= Z_x\cdot Z_y, \\ e_\psi &= |(Z\circ\psi)_u|^2, & f_\psi &= |(Z\circ\psi)_v|^2, & g_\psi &= (Z\circ\psi)_u\cdot(Z\circ\psi)_v. \end{aligned}$$

Then, see Exercise 4.8, we find

$$e_\psi f_\psi - g_\psi^2 = \Big(ef - g^2\Big)\circ\psi\left(\det(\nabla\psi)\right)^2. \tag{4.18}$$

Note that since ∇Z is non-degenerate, $ef - g^2 > 0$, and hence $ef - g^2 \in C^1(\overline{D})$. Therefore, by Lemma 4.11, we conclude

$$A(Z\circ\psi) = \int_D \sqrt{e_\psi f_\psi - g_\psi^2} = \int_D \sqrt{\Big(ef - g^2\Big)\circ\psi}\, \det(\nabla\psi) = A(Z).$$

This completes the proof of the lemma. □

We can now prove the main result of this section.

Theorem 4.5 *Let Γ be a simple closed Jordan curve of class C^2. Then*

$$\inf_{\mathcal{H}_\Gamma} E = \inf_{\mathcal{H}_\Gamma} A.$$

Proof. It suffices to show that

$$\inf_{\mathcal{H}_\Gamma} E \leq \inf_{\mathcal{H}_\Gamma} A \tag{4.19}$$

Let $X \in \mathcal{H}_\Gamma \cap C^2(\overline{D}; \mathbb{R}^n)$. For $\varepsilon > 0$, define $X_\varepsilon \in C^2(\overline{D}; \mathbb{R}^{n+2})$, by

$$X_\varepsilon(\zeta) = \big(X(\zeta), \varepsilon\zeta\big).$$

For each $\varepsilon > 0$, we claim that there exists $\psi \in \overline{\mathcal{F}}$, such that

$$A(X_\varepsilon) = E(X_\varepsilon \circ \psi).$$

Clearly,

$$E(X_\varepsilon \circ \psi) \geq E(X \circ \psi) \geq \inf_{\mathcal{H}_\Gamma} E,$$

and, by continuity of A, we have

$$A(X_\varepsilon) \to A(X),$$

as $\varepsilon \to 0$. Thus, $A(X) \geq \inf_{\mathcal{H}_\Gamma} E$, and taking the infimum over all $X \in \mathcal{H}_\Gamma \cap C^2(\overline{D}; \mathbb{R}^n)$, we obtain

$$\inf_{\mathcal{H}_\Gamma} E \leq \inf_{\mathcal{H}_\Gamma \cap C^2(\overline{D}; \mathbb{R}^n)} A.$$

Inequality (4.19) now follows from the density of $C^2(\overline{D}; \mathbb{R}^n)$ in $\mathcal{H}_\Gamma$, and the continuity of A. To prove the claim, fix $\varepsilon > 0$, and consider the functional

$$F(\psi) = E(X_\varepsilon \circ \psi)$$

on $\overline{\mathcal{F}}$. We have

$$F(\psi) \geq \varepsilon E(\psi),$$

hence F is coercive on $\overline{\mathcal{F}}$. If $\psi_k \rightharpoonup \psi$ in $H_1(D; \mathbb{R}^2)$, then $X_\varepsilon \circ \psi_k \rightharpoonup X_e \circ \psi$ in $H_1(D; \mathbb{R}^{n+2})$, therefore F weakly lower semi-continuous. Clearly $\overline{\mathcal{F}}$ is weakly closed, therefore, by Theorem 3.6, there exists a minimizer, i.e. $\psi_0 \in \overline{\mathcal{F}}$ such that

$$E(X_\varepsilon \circ \psi_0) \leq E(X_\varepsilon \circ \psi),$$

for all $\psi \in \overline{\mathcal{F}}$. It follows, as in the proof of Theorem 4.1, that $X_\varepsilon \circ \psi_0$ is conformal, and thus

$$A(X_\varepsilon \circ \psi_0) = E(X_\varepsilon \circ \psi_0).$$

X_ε is non-denegerate, thus by Lemma 4.12, $A(X_\varepsilon \circ \psi_0) = A(X_\varepsilon)$, and the claim follows. This completes the proof of the theorem. $\square$

We now restate our existence result.

Theorem 4.6 *Let Γ be of class C^2. Then there exists a solution of Plateau's Problem, i.e. a conformally parameterized minimal surface of least area among all surfaces of disk type spanning Γ.*

4.4 Exercises

Exercise 4.1 Show that A is continuous on $\mathcal{H}_\Gamma$.

Exercise 4.2 Let $\tau \in C^1(\overline{D}; \mathbb{R}^2)$. Show that if $\sup_D |\nabla\tau|\,|t| < 1$, then $g_t = \mathrm{id} + t\tau$ is a diffeomorphism.

Exercise 4.3 Verify Equation (4.5).

Exercise 4.4 Prove Lemma 4.5

Exercise 4.5 Let X and Y be compact Hausdorff spaces, and let $f\colon X \to Y$ be a continuous monotone map onto Y, i.e. f is continuous and $f^{-1}(y)$ is connected for every $y \in Y$. Show that $f^{-1}(K)$ is connected for every compact connected $K \subset Y$.

Exercise 4.6 Let $\sigma_k\colon S^1 \to S^1$ be a sequence of continuous monotone maps, and suppose that $\sigma_k \to \sigma$ pointwise, and σ is continuous. Then σ is monotone.

Exercise 4.7 Let V and W be finite dimensional inner-product vector spaces, V^* and W^* respectively their duals, and identify $V \cong V^*$, $W \cong W^*$, via the inner-product, e.g.

$$v \mapsto (\,\cdot\,, v)_V,$$

for $v \in V$. Let $T\colon V \to W$ be a linear transformation, $T^*\colon W^* \to V^*$ its adjoint, then $T^*T\colon V \to V$, and we define $\det(T^2) = \det(T^*T)$. Let $A\colon V \to V$ be a linear transformation. Show that

$$\det\Big((TA)^2\Big) = \det(T^2)\Big(\det(A)\Big)^2.$$

Exercise 4.8 Use Exercise 4.7 to prove equation (4.18)

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